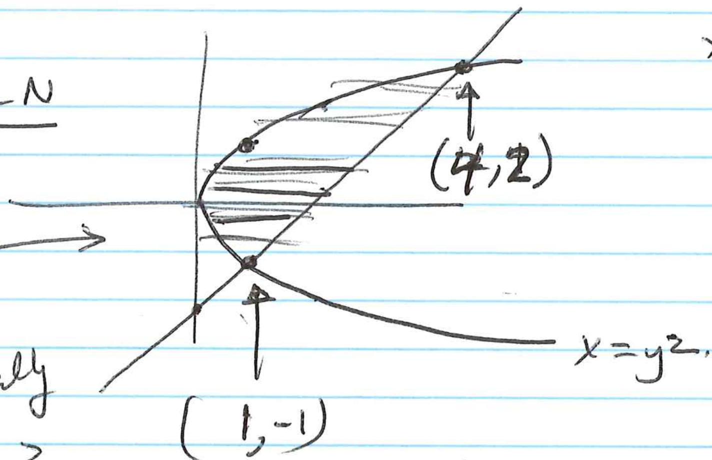
§15.3

WARM UP 15.2.15 (1)

$\iint_D y\, dA$ D bounded by $y = x-2$ $x = y+2$

$x = y^2$

SOLN



meet: $y^2 = y+2$

$y^2 - y - 2 = 0$

$(y-2)(y+1)$

$y = 2, -1$

$$\int_{y=-1}^{2}\int_{x=y^2}^{y+2} y\,dx\,dy = \int_{-1}^{2}\left(\Big|_{y^2}^{y+2} xy\right) dy$$

$$= \int_{-1}^{2}(y^2+2y-y^3)\,dy = \Big|_{-1}^{2} \frac{y^3}{3}+y^2-\frac{y^4}{4} = \frac{9}{4}$$

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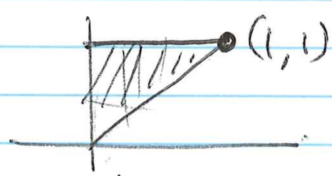
~~§15.3 Problem~~

WARM UP 2

$$\int_{x=0}^{1}\int_{y=x}^{1} \sin(y^2)\,dy\,dx$$ ← seems impossible!

(1) Draw Region:

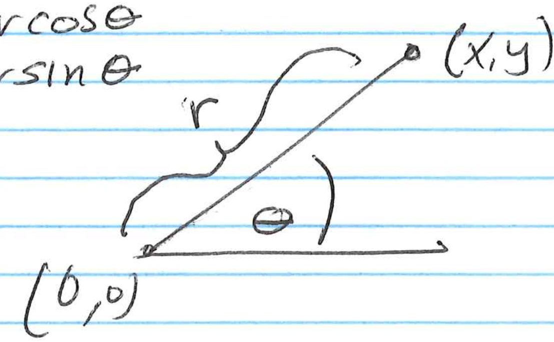


(2) Switch order: $\int_{y=0}^{1}\int_{x=0}^{y} \sin(y^2)\,dx\,dy$

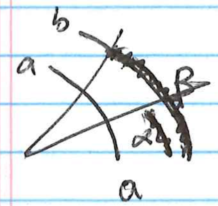
$$= \int_{y=0}^{1} y\sin y^2\,dy = \Big|_0^1 -\frac{\cos(y^2)}{2}$$ ///

§15.3 Polar

<u>recall</u> $x = r\cos\theta$
$y = r\sin\theta$



Book calls $a \le r \le b$, $\alpha \le \theta \le \beta$ a "polar rectangle"



<u>Key</u>: chain rule

$$dx = -r\sin\theta\, d\theta + \cos\theta\, dr$$
$$dy = r\cos\theta\, d\theta + \sin\theta\, dr$$

So

$$dA = dx\,dy = -r^2\cos\theta\sin\theta\, d\theta^2 + r\cos^2\theta\, dr\,d\theta + r\sin^2\theta\, dr\,d\theta + \cos\theta\sin\theta\, dr^2$$

as $dr, d\theta$ shrink to zero $dr^2, d\theta^2 \to 0$ <u>fast</u>

leaving $r(\underbrace{\cos^2\theta + \sin^2\theta}_{=1})\,dr\,d\theta$

We'll see another reason when we do §15.9

$\boxed{\text{So}}$ $dx\,dy \xRightarrow{\text{replace with}} r\,dr\,d\theta$

hardest part of Calc III

<u>Reality Check</u>: $\displaystyle\iint_{\text{unit circle}} 1\, dx\,dy = \pi$

(?)

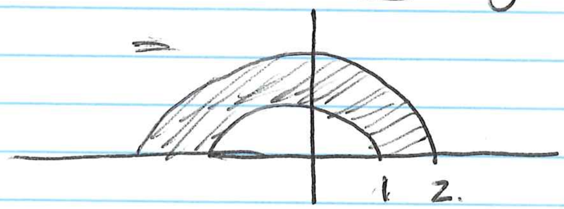
$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r\,d\theta\,dr = 2\pi\int_{r=0}^{1} r\,dr = 2\pi \left.\frac{r^2}{2}\right|_0^1 = 2\pi\cdot\frac{1}{2} = \pi$$

////

Example (me)

$$\iint_D 3x+4y^2\,dA,\quad D = \text{Area between } x^2+y^2=1,\ x^2+y^2=4 \text{ and } y \geq 0$$

=



$$= \int_{r=1}^{2}\int_{\theta=0}^{\pi}\left(3r\cos\theta + 4r^2\sin^2\theta\right) r\,dr\,d\theta$$

$\Rightarrow$ split up $3\int_1^2\int_0^\pi r^2\cos\theta\,d\theta\,dr = 3\int_1^2 r^2\,dr \underbrace{\left.\sin\theta\right|_0^\pi}_{0} = 0$

$$+\ 4\int_1^2\int_0^\pi r^3\sin^2\theta\,d\theta\,dr. \qquad \text{TRIG ID}\quad \sin^2\theta = \frac{1-\cos2\theta}{2}$$

$$= 2\int_1^2\int_0^\pi r^3\underbrace{(1-\cos2\theta)\,d\theta}_{\left.\theta-\frac{\sin2\theta}{2}\right|_0^\pi}\,dr$$

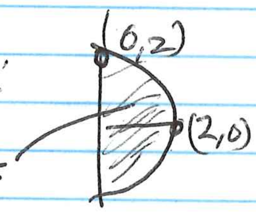
$$2\int_1^2 r^3\cdot\pi\,dr = 2\pi\left.\frac{r^4}{4}\right|_1^2 = \frac{30\pi}{4}$$

15.3 ③

Class

$$\iint e^{-x^2-y^2}\,dA, \quad D \text{ bounded by } x=\sqrt{4-y^2} \text{ and } y\text{-axis}$$

SOLN: D



$$= \int_{y=-2}^{2}\int_{x=0}^{\sqrt{4-y^2}} e^{-x^2-y^2}\,dx\,dy$$

CAN'T DO!

POLAR: $r = 0..2$
$\theta = -\pi/2 .. \pi/2$

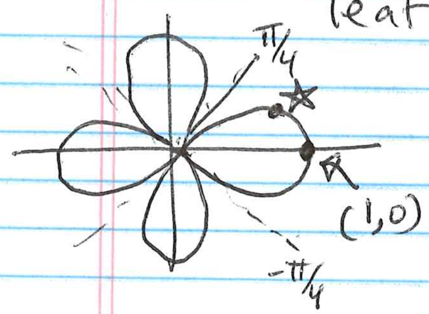
$$\int_{r=0}^{2}\int_{\theta=-\pi/2}^{\pi/2} e^{-r^2}\, r\,dr\,d\theta$$

$$= \pi\int_{r=0}^{2} e^{-r^2}\, r\,dr = -\frac{\pi}{2}\int_0^2 e^{-r^2}(-2r)\,dr$$

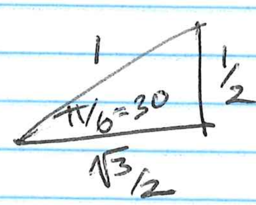
$$= -\frac{\pi}{2}\Big|_0^2 e^{-r^2} = -\frac{\pi}{2}\left(e^{-4}-1\right)$$

Class ((example 3 in book))

Find area of one closed leaf of $r = \cos 2\theta$



First - suppose you didn't have picture: Plot points!



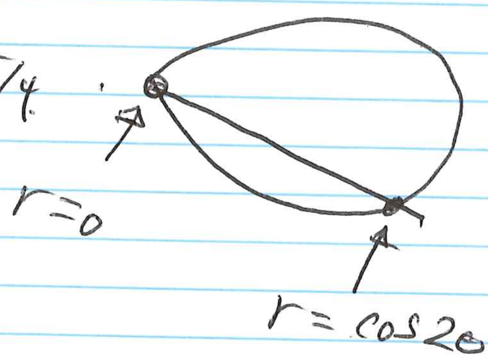
θ	r	(x,y)
0	1	(1,0)
$\pi/12$	$\frac{\sqrt{3}}{2}$	☆

15.3

So we figure out if we let θ run from $-\pi/4$ to $\pi/4$ we get one petal.

$$\iint_{\text{petal}} dx\,dy = \iint_{\text{lims ?}} r\,dr\,d\theta$$

$\theta = -\pi/4$ to $\pi/4$.



$$\int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{\cos 2\theta} r\,dr\,d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} (\cos 2\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1+\cos 4\theta}{2}\,d\theta$$

$$\cos^2(x) = \frac{\cos 2x + 1}{2}.$$

$$= \frac{1}{4}\Big[\theta + \frac{\sin 4\theta}{4}\Big]_{-\pi/4}^{\pi/4} = \frac{1}{4}\left(\frac{\pi}{2}\right) = \boxed{\pi/8}$$