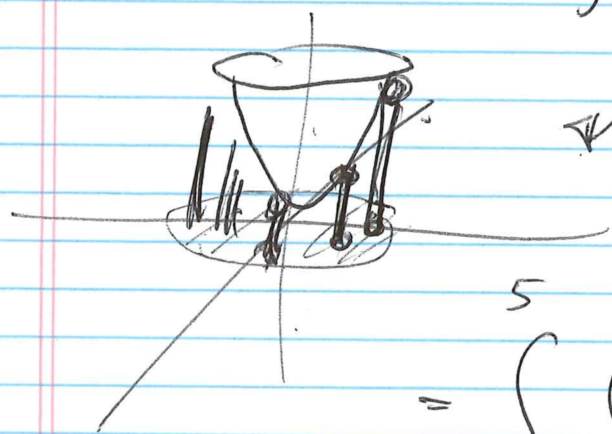
§ 15.5
Surface Area

WARM UP 15.3.19

Find Volume under paraboloid $z = x^2 + y^2$

above the disk $x^2 + y^2 \le 25$ $\boxed{\text{Use polar}}$



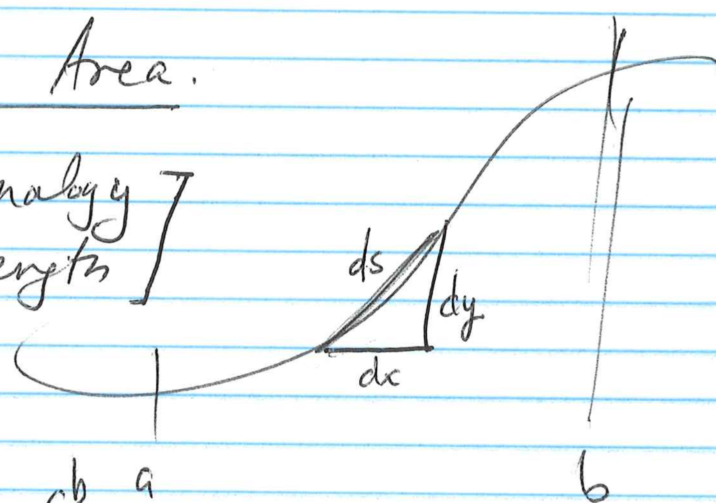
height = roof.

$$\iint_{\text{Disk}} (x^2+y^2)\, dx\, dy$$

$$= \int_{r=0}^{5} \int_{\theta=0}^{2\pi} r^2 \cdot r \, d\theta \, dr = 2\pi \Big|_0^5 \frac{r^4}{4} = \frac{5^4 \pi}{2}$$

§ 15.5 Surface Area.

[Basic analogy is to arclength]

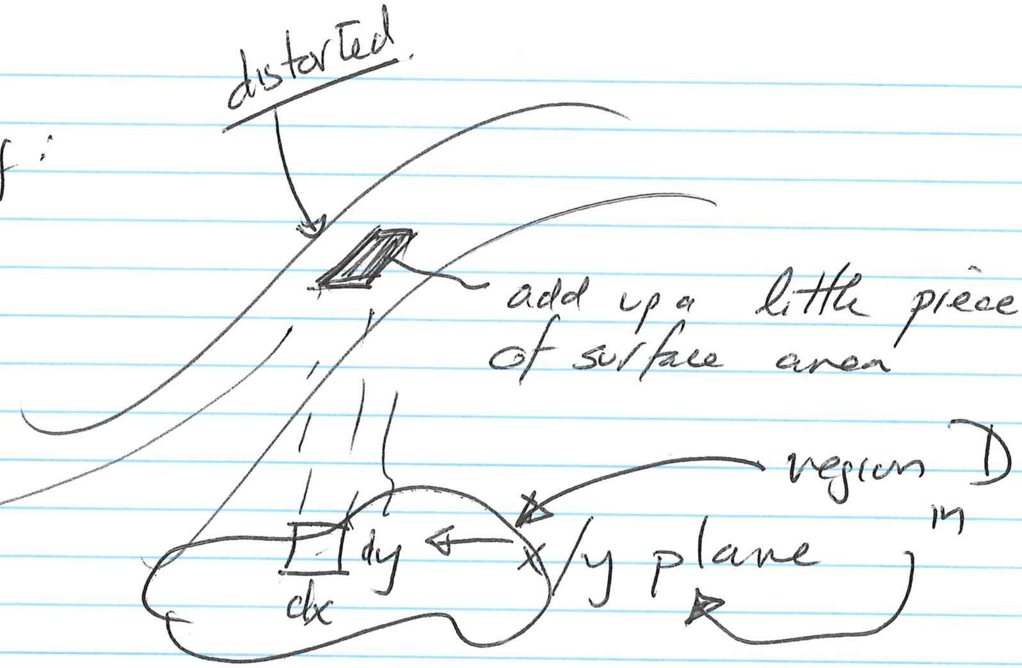


$ds^2 = dx^2 + dy^2$.

$\Rightarrow ds = \sqrt{(x')^2 + (y')^2} \Rightarrow \int_a^b \sqrt{(x')^2 + (y')^2}\, dt.$

15.5:

Analog:



idea it is just like what we did to get the normal vector to a surface!

area is $|\vec{v} \times \vec{w}|$, where ~~v×w~~

$v = \langle -f_x, 0, 1 \rangle$ $\quad \vec{v} \times \vec{w} = \langle -f_x, -f_y, 1 \rangle$

$w = \langle 0, -f_y, 1 \rangle$

$$\boxed{\text{so } |v \times w| = \sqrt{f_x^2 + f_y^2 + 1}\ dx\,dy}$$

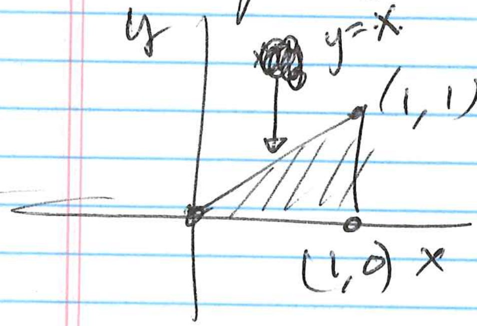
AND Surface Area $= \iint_D \sqrt{f_x^2 + f_y^2 + 1}\ dx\,dy$

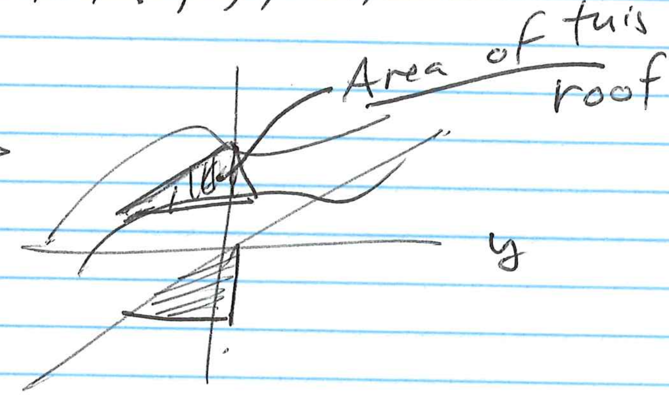
$\boxed{\text{if } z = f(x,y)}$

Example 1 (Me)

$z = x^2 + 2y$

above triangle w/ vertices $(0,0), (1,0), (1,1)$.

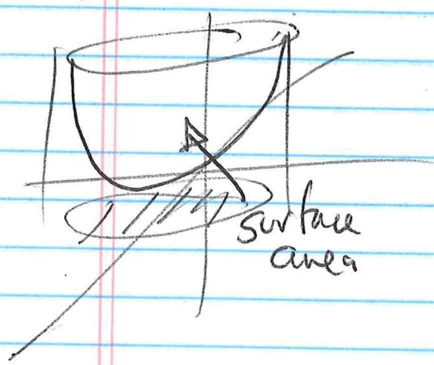




$$= \int_{x=0}^{1} \int_{y=0}^{x} \sqrt{\underbrace{(2x)^2 + (2)^2 + 1}_{4x^2+5}}\; dy\, dx.$$

$$= \frac{1}{8} \int_{x=0}^{1} \underbrace{8x}_{du} \sqrt{\underbrace{4x^2+5}_{u}}\; dx = \frac{1}{8} \Big|_{0}^{1} \frac{2}{3}(4x^2+5)^{3/2} = \frac{1}{12}\left(9^{3/2} - 5^{3/2}\right)$$

Class Find area of paraboloid $z = x^2 + y^2$ under plane $z = 9$.



$z = 9 \Rightarrow$ circle of radius 3 = base in x–y plane

Hence, we want $\displaystyle\iint_{\substack{\text{circle}\\\text{of}\\\text{radius 3}}} \sqrt{(2x)^2+(2y)^2+1}\,dx\,dy.$

This screams! Polar

take out

$$= \frac{1}{8}\int_{\theta=0}^{2\pi}\int_{r=0}^{3} \sqrt{4r^2+1}\,\underbrace{r\,dr}_{8}\,d\theta$$

put in

$$= \frac{1}{8}\int_{0}^{2\pi} \left. (4r^2+1)^{3/2}\cdot\frac{2}{3}\right|_{r=0}^{3} d\theta \qquad u=(4r^2+1)^{\frac{1}{2}} \quad du = 8r\,dr$$

$$= \frac{1}{12}\int_{0}^{2\pi}\left(37^{3/2}-1\right)d\theta = \frac{\pi}{6}\left(37^{3/2}-1\right)$$

(13) Find Area of Part of surface $z = \dfrac{1}{1+x^2+y^2}$ above disk $x^2+y^2 \le 1$.

$$\iint_{\substack{\text{unit}\\\text{disk}}} \sqrt{\left(\frac{2x}{(1+x^2+y^2)^2}\right)^2+\left(\frac{2y}{(1+x^2+y^2)^2}\right)^2+1}\;dx\,dy.$$

Polarize

$$\int_{r=0}^{1}\int_{\theta=0}^{2\pi} \frac{4x^2+4y^2+(1+x^2+y^2)^4}{(1+x^2+y^2)^4}\,dx\,dy$$

$$= \iint \frac{4r^2\ (1+r^2)^4}{(1+r^2)^4}\,r\,dr\,d\theta$$