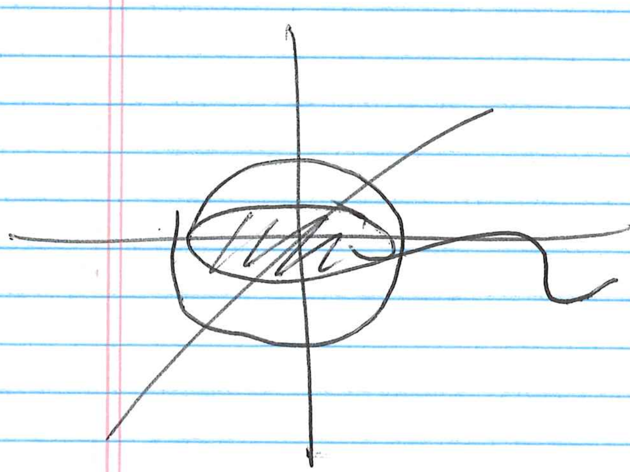
§15.6

WARM UP Use double integrals to find volume of a sphere of radius a

$x^2+y^2+z^2=a^2$

$\Rightarrow z=\sqrt{a^2-x^2-y^2} \leftarrow +/-$



R = circle of radius a in x-y plane.

roof / floor = +z and −z

So $$\iint_{\text{circle radius } a} \underbrace{\overbrace{\sqrt{a^2-x^2-y^2}}^{\text{roof}} - \overbrace{(-)\sqrt{a^2-x^2-y^2}}^{\text{floor}}}_{2\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

circle → radius a

POLARIZE!

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{a} 2\sqrt{a^2-r^2}\;r\,dr\,d\theta$$

$$\boxed{\begin{array}{l} u=a^2-r^2 \\ du=-2r\,dr\,d\theta \\ \int u^{1/2}=\tfrac{2}{3}u^{3/2} \end{array}}$$

$$= -\int_{\theta=0}^{2\pi} \frac{2}{3}\Big|_0^a (a^2-r^2)^{3/2}\,d\theta$$

$$= \frac{2}{3}(a^2)^{3/2}\int_0^{2\pi} d\theta = \frac{4\pi}{3}(a^2)^{3/2} = \boxed{\frac{4}{3}\pi a^3}$$

§ 15.6 : $\iiint$

It works *exactly* like double integrals, but with one more parameter:

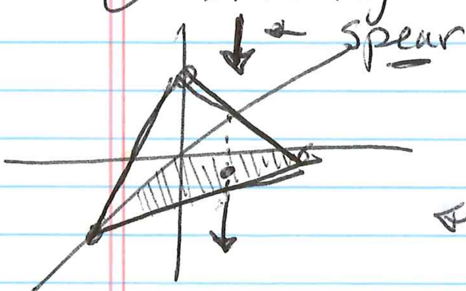
Example 1: $\displaystyle\int_{x=0}^{1}\int_{y=-1}^{2}\underbrace{\int_{z=0}^{3} z^2\,dz}_{\left.\frac{z^3}{3}\right|_0^3}\,dy\,dx$

$\displaystyle\int_{x=0}^{1}\int_{y=-1}^{2} 9\,dy\,dx \leftarrow$ double int, done by earlier work on $\iint$.

$\boxed{\text{What is }{}^{*}\text{hard}^{*}}$: Setting limits on complicated regions

Example 2: $\displaystyle\iiint_R z\,dV$ (volume $= V$) where R is bounded by coord planes + plane $x+y+z=1$

① Draw region if you can.

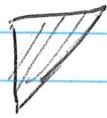


② Pick "innermost" integral, Throw spear along that axis

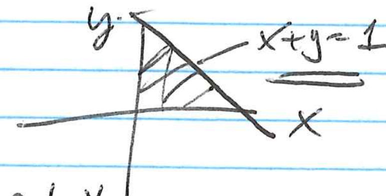
In example, if we make dz the innermost, spear hits "roof" at $z=1-x-y$, exits "floor" at $z=0$.

$$\iint_R \int_{z=0}^{1-x-y} z\,dz\,dx\,dy$$

R 

The $\iint$  part comes from where "spear" hits x-y plane when $z=0$

$$x+y+z=1 \quad \& \quad z=0 \implies$$

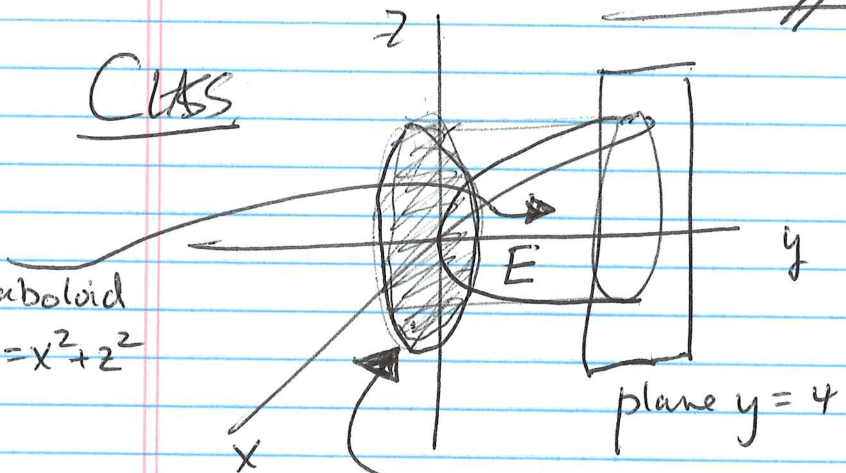


So outer integral is $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x}$ stuff

$$\left[\begin{array}{c}\text{WHOLE}\\ \text{THING}\end{array}\right] \implies \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} z\,dz\,dy\,dx \quad \left\{\begin{array}{c}\text{easy to}\\ \text{do}\end{array}\right\}$$

do at home $\left(\frac{1}{24}\right)$ = ans.

CLASS



$$\iiint_E \sqrt{x^2+z^2}\,dV$$

Spear along y axis, shadow in x-z plane is

$4=x^2+z^2$ circle radius 2

$$\iint \int_{y=x^2+z^2}^{4} \sqrt{x^2+z^2}\,dV$$

Polar! $\displaystyle\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{y=r^2}^{4} \sqrt{r^2}\,dy\,r\,dr\,d\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r\cdot(4-r^2)\,r\,dr\,d\theta$$

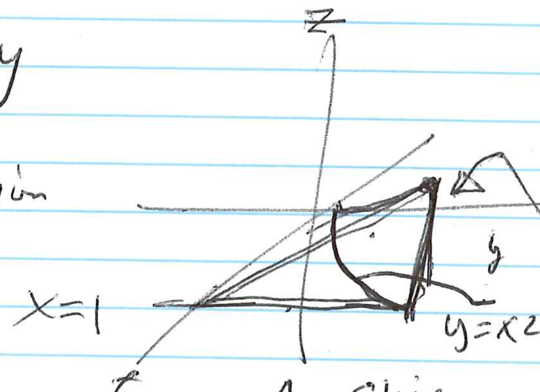
$$= \int_0^{2\pi} \underbrace{\int_0^2 4r^2 - r^4 \, dr}_{\frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2} = \left(\frac{32}{3} - \frac{32}{5}\right) 2\pi = \left(\frac{128\pi}{15}\right)$$

//

$\boxed{\text{Class}}$ Example 4 $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dz\, dy\, dx$

rewrite $\iiint f \, dx\, dz\, dy$

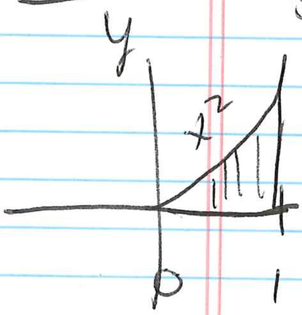
Soln: have to draw region



A slice of cheese, Someone took a bite.

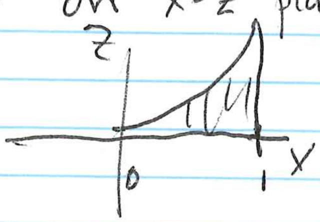
~~as z goes 0..y~~

OUTERMOST ~~y goes 0..x^2.~~



on x-y plane

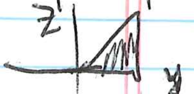
on x-z plane



$z = 0..y \leftarrow 0..x^2$.

$$\Rightarrow \underbrace{\iiint}_{\substack{x=\sqrt{y} \\ (y=x^2)}}^{1}$$

in y-z plane



$y = 0..x^2 \leftarrow$ goes to 1 SHADOW

$z = 0$ to y

in y/z

$$= \int_{y=0}^{1} \int_{z=0}^{y} \int_{x=\sqrt{y}}^{1} f(x,y,z)\, dx\, dz\, dy.$$