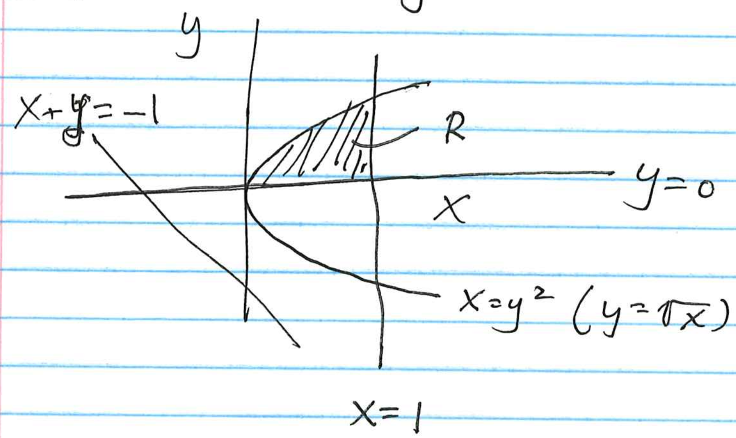
15.7 - 15.8
Cylindrical + Spherical Coords

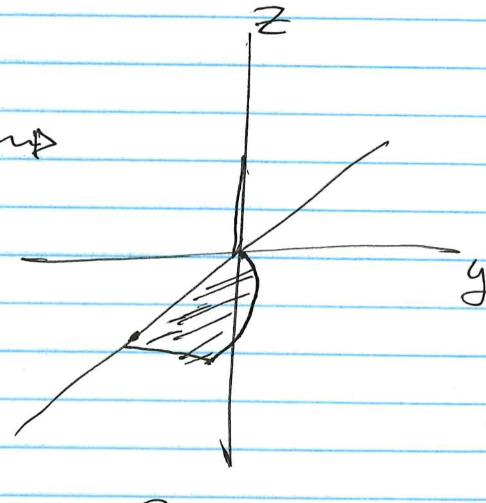
WARM UP: 15.6.13 Compute $\iiint_E 6xy\, dV$

where E is region under plane $z = 1+x+y$ and above region in x-y plane bounded by curves $y = \sqrt{x}$, $y = 0$, $x = 1$.

SOLN Draw region in x-y plane:



$\Rightarrow$



where does $z = 1+x+y$ hit x-y plane? A: $z=0$
$\Rightarrow x+y=-1$

$$\Rightarrow \iint_R \int_{z=0}^{1+x+y} 6xy\, dz\, dx\, dy.$$

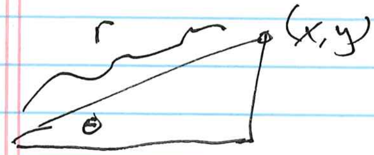
$$= \iint_R 6xy\,(1+x+y)\, dx\, dy.$$ Limits for R: $x = 0..1$, $y = 0..\sqrt{x}$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{x}} 6xy + 6x^2y + 6xy^2\, dy\, dx = \int_{x=0}^{1} \Big|_{y=0}^{\sqrt{x}} 3xy^2 + 3x^2y^2 + 2xy^3$$

$$= \int_{x=0}^{1} 3x^2 + 3x^3 + 2x^{5/2} = \Big|_0^1 x^3 + \frac{3x^4}{4} + \frac{4}{7}x^{7/2} \quad 1 + \frac{3}{4} + \frac{4}{7} = 65/28$$

§ 15.7 Cylindrical Coords (easy ⇒ just polar)

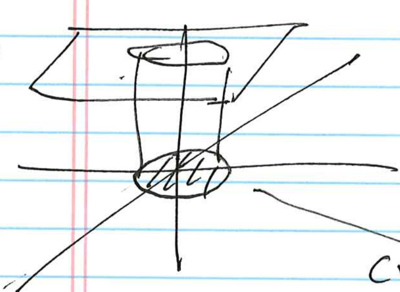
$x = r\cos\theta$
$y = r\sin\theta$ } polar in x-y plane
$z = z$ $\qquad z = z$ } **NOTICE** Can also do with polar in x-z or y-z plane



$r^2 = x^2 + y^2$, $\tan\theta = y/x$.

Example [me] Use triple integral to find volume below plane $z = 4$, inside circle $x^2+y^2=1$

Z = 4



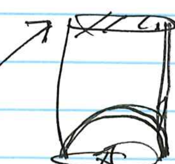
Circle $r=1$

Soln: $\displaystyle\iint\int_{z=0}^{4} dz\,dx\,dy$

$\displaystyle = \iint 4\,dx\,dy = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} 4\,d\theta\, r\,dr = \boxed{4\pi}$

[Class] What if below plane $z=4$
above surface $z = 1-x^2-y^2$.

Flat top



base is "pushed in".

Same $\displaystyle\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=1-r^2}^{4} dz\,d\theta\,r\,dr = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} 3+r^2\,d\theta\,r\,dr$

$\displaystyle = 2\pi\int_{r=0}^{1} 3r+r^3\,dr$

$\displaystyle = 2\pi \left.\frac{3r^2}{2}+\frac{r^4}{4}\right|_0^1 = 2\pi\cdot 7/4$

$= \boxed{7\pi/2}$

Class (Reps 1) Setup (and evaluate on your own)

(A) $\iiint_E \sqrt{x^2+y^2}\,dV$, E = inside $x^2+y^2=16$, $z=-5$ to 4

(B) $\iiint_E x^3+xy^2\,dV$, E= First octant, undr $z=1-x^2-y^2$

(C) $\iiint_E e^z\,dV$, E = under paraboloid $z=1+x^2+y^2$, above xy plane (so $z \geq 0$) inside $x^2+y^2=5$.

TAKE 10-15 minutes to do these

(A) $$\int_{\theta=0}^{2\pi}\int_{r=0}^{4}\int_{z=-5}^{4} r\,dz\,r\,dr\,d\theta \left(= 9\iint r^2\,dr\,d\theta = 9\cdot\frac{r^3}{3}\Big|_0^4 \cdot 2\pi = \underline{4^3\cdot 6\pi}\right.$$

(B) $x,y,z \geq 0$, $z \leq 1-r^2$ $$= \iiint_{z=0}^{1-r^2} \underbrace{r^2}_{x^2+y^2}\cdot\overbrace{r\cos\theta}^{x}\cdot dz\cdot r\,dr\,d\theta$$

to get these, it is when $z=0 \Rightarrow$ $1-r^2=0 \Rightarrow r=0..1$

First quadrant $\Rightarrow$ $\theta=0..\pi/2$

$$\Rightarrow \int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\int_{z=0}^{1-r^2} dz\, r^4\cos\theta\,dr\,d\theta = \boxed{\frac{1}{30}}$$

(C) x-y limits are circle of radius $\sqrt{5}$.

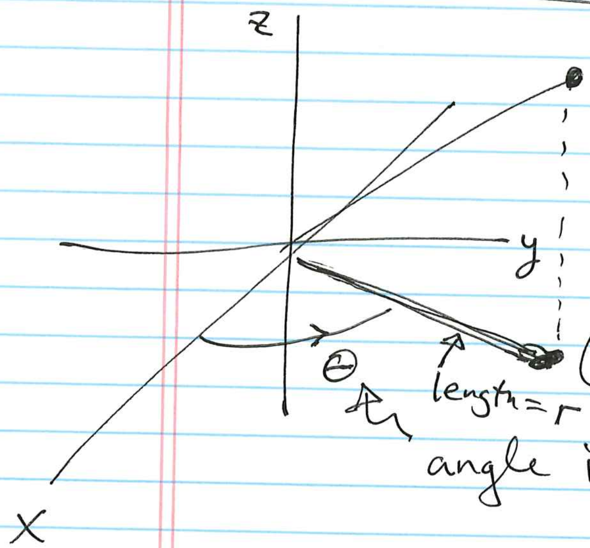
$$\int_{r=0}^{\sqrt{5}}\int_{\theta=0}^{2\pi}\int_{z=0}^{1+r^2} e^z\,dz)\,d\theta\,r\,dr$$

$$= \int_{r=0}^{\sqrt{5}}\int_{\theta=0}^{2\pi}\left(e^{1+r^2}-1\right)d\theta\,r\,dr = \pi(e^6-e-5)$$

OK, easy part done

of 15.8 spherical

15.8 $\boxed{\text{KEY PART OF TODAY}}$



(x,y,z) point in space $= p$

$\rho =$ distance from origin to p

$= \sqrt{x^2+y^2+z^2}$.

drop p to x-y plane = set z coord to 0

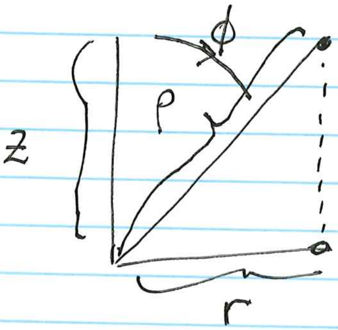
$(x,y,0) = p'$

angle in x-y plane between x-axis and p'

$\phi =$ angle between z-axis and p

So, we get $p' = (x,y,0) = (r\cos\theta, r\sin\theta, 0)$.

KEY ?: How do we relate ρ, ϕ, r, θ ?



From picture, we see that

- $\sin\phi = r/\rho \Rightarrow r = \rho\sin\phi$
- $\cos\phi = z/\rho \Rightarrow z = \rho\cos\phi$

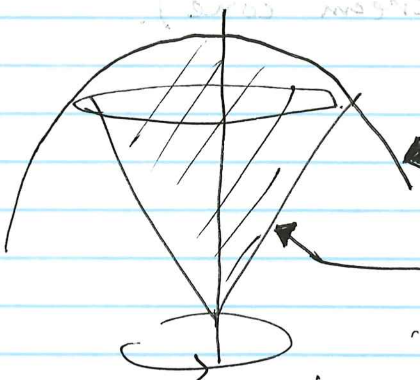
$\boxed{\text{Put it all together}}$

$$\left.\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta \\ y &= r\sin\theta = \rho\sin\phi\sin\theta \\ z &= \rho\sin\phi \end{aligned}\right\}$$

☆ TATOO ON YOUR BRAIN!

An easy (but messy) chain rule computation shows $\boxed{dx\,dy\,dz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$

$\boxed{\text{Example 4 (book)}}$ Find volume above $z=\sqrt{x^2+y^2}$ (45° cone) and below sphere $x^2+y^2+z^2=z$.



θ spins whole way around $\Rightarrow \theta = 0..2\pi$

Soln: notice our cone has $\phi = 0..\pi/4$ and $\rho = 0$ to surface of sphere

$\rho^2 = x^2+y^2+z^2 = z = \rho\cos\phi$

$\rho = \cos\phi$. so $\underline{\underline{\rho = \cos\phi}}$

$\Rightarrow$ integral is $\displaystyle\int_{\phi=0}^{\pi/4}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{}$

$dx\,dy\,dz = \rho^2\sin\phi \, d\rho \, d\theta \, d\phi$

$= \Big|_{\rho=0}^{\cos\phi} \dfrac{\rho^3}{3} = \dfrac{\cos^3\phi}{3}$

$\Rightarrow \dfrac{1}{3}\displaystyle\int_{\phi=0}^{\pi/4}\int_{\theta=0}^{2\pi} \cos^3\phi \sin\phi \, d\theta \, d\phi = \dfrac{2\pi}{3}\int_0^{\pi/4} \cos^3\phi\sin\phi \, d\phi$

$= \dfrac{2\pi}{3}\Big|_0^{\pi/4} -\dfrac{\cos^4\phi}{4}$

$= \dfrac{2\pi}{12}\left(-\underbrace{\dfrac{\sqrt{2}}{2}^4 + 1}\right)$

$\dfrac{2\pi}{12}\cdot \dfrac{3}{4} = \boxed{\dfrac{\pi}{8}}$

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Class (Reps!)

Ⓐ Change $(-1, 1, -\sqrt{2})$ to spherical
x, y, z

Ⓑ Change $(3, \pi/2, 3\pi/4)$ to (x, y, z)
(ρ, θ, ϕ)

Ⓒ Evaluate $\displaystyle\iiint_E \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}\,dV$, E = First octant and $x^2+y^2+z^2 \le 1$

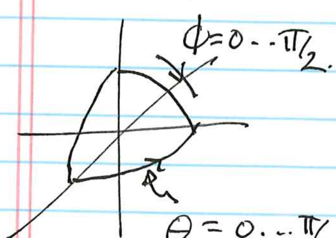
Take 10-15 minutes to do them

Ⓐ $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{4} = 2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{3\pi}{4}$, $\phi = \sin^{-1}\left(\frac{z}{\rho}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 3\pi/4.$

Ⓑ
$$\left.\begin{aligned}
x &= \rho\sin\phi\cos\theta = 3\sin 3\pi/4 \cos \pi/2 = 0\\
y &= \rho\sin\phi\sin\theta = 3\sin 3\pi/4 \sin\pi/2 = 3\cdot\frac{\sqrt{2}}{2}\\
z &= \rho\cos\phi = 3\cos\tfrac{3\pi}{4} = 3\cdot -\frac{\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}
\end{aligned}\right\}.$$

Ⓒ Limits are easy : $x, y, z \ge 0$ (first octant) and $\rho^2 \le 1 \Rightarrow \rho \le 1$

in spherical, $\rho = 0..1$, $\phi = 0..\pi/2$, $\theta = 0..\pi/2$.



$$\Rightarrow \iiint \underbrace{e^{\rho^2}}_{e^{x^2+y^2+z^2}} \cdot \underbrace{\frac{1}{\rho}}_{\sqrt{x^2+y^2+z^2}} \cdot \underbrace{\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}_{dV}$$

first, $\int$ out $\sin\phi = -\cos\phi\Big|_{\phi=0}^{\pi/2} = \boxed{1}$

next, $\int$ out $d\theta = \boxed{\pi/2}$

last $\displaystyle\int_{\rho=0}^{1} e^{\rho^2}\cdot\frac{1}{\rho}\cdot\rho^2\,d\rho = \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}\Big|_0^1 e^{\rho^2} = \frac{1}{2}(e-1)$

$u = \rho^2$
$du = 2\rho\,d\rho$

Whole answer is

$1\cdot\frac{\pi}{2}\cdot\frac{1}{2}(e-1)$

$\boxed{= \frac{\pi(e-1)}{4}}$