15.9

HARDEST DAY OF CLASS: $\triangle$ COORDS

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WARM UP:

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \cancel{\rho \sin}\ \rho\cos\phi$

15.8. 23. $\iiint_E (x^2+y^2)\,dV$, $E =$ region between spheres of radius ~~2~~ and ~~3~~.

Lims are easy: $\phi = 0..\pi$, $\theta = 0..2\pi$, $\rho = 2..3$.

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=2}^{3} \underbrace{(\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)}_{\rho^2\sin^2\phi}\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$x^2+y^2 =$

$$\underbrace{\int_2^3 \rho^4\, d\rho}_{} = \rho^5/5\Big|_2^3 = \frac{3^5-2^5}{5}$$ constant, now pull it thru

$$= \left(\frac{3^5-2^5}{5}\right)\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin^3\phi\, d\phi\, d\theta$$ swap order, get rid of θ

$$= 2\pi\left(\frac{3^5-2^5}{5}\right)\int_{\phi=0}^{\pi}\sin^3\phi\, d\phi$$ good trick $\sin^2\phi = 1-\cos^2\phi$

$$= 2\pi\left(\frac{3^5-2^5}{5}\right)\int_0^{\pi}\underbrace{(1-\cos^2\phi)\sin\phi}\, d\phi$$

$$= \int \sin\phi - \int \cos^2\phi\sin\phi$$

$$= \underbrace{\Big|_0^{\pi} -\cos\phi}_{2} + \underbrace{\Big|_0^{\pi}\frac{\cos^3\phi}{3}}_{-2/3}$$

$$\boxed{= 2\pi\left(\frac{3^5-2^5}{5}\right)\left(\frac{4}{3}\right)}$$

§ 15.9 Δ - coords

We're been doing this already when we go $(x,y) \longmapsto (r,\theta)$ polar

or $(x,y,z) \longmapsto (\rho,\theta,\phi)$ spherical

Three steps

(1) swap the vars. For example, if

$\int f(x,y)$ and $x = g(s,t)$
$y = h(s,t)$

(EASY) then $\int f(x,y) \rightarrow \int f(g(s,t), h(s,t))$

(2) since $x = g(s,t)$
$y = h(s,t)$, $dx\,dy$ changes to

Q: who is this? $\longrightarrow$ $\boxed{\text{something}} \cdot ds\,dt$

(SEMI-EASY) A: Jacobian matrix, measures "distortion"

Algorithm make a matrix of partials, take determinant.

Example (polar) $x = r\cos\theta$
$y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

"distortion" = |Jacobian matrix|

absolute value of

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$\boxed{= r}$$

this is why

$dx\,dy \longmapsto r\,dr\,d\theta$!!

Exercise for masochists

Do this for
$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

you'll get a 3×3 matrix, which has determinant equal to $\rho^2 \sin\phi$!

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OK, I said 3 parts, (1) is easy and (2) is (semi) easy. Whats hard?

CHANGING LIMITS

Flashback to calc II and ~~the~~ u-substitution.

$x = g(u)$
$$\int_a^b f(x)\,dx = \int_{?}^{?} \underbrace{f(g(u))}_{\text{PART 1}} \underbrace{g'(u)}_{\text{PART 2}}\,du$$

WHO GOES HERE = PART 3.

If $a = g(c)$
$b = g(d)$, then the answer is $\displaystyle\int_{c = g^{-1}(a)}^{d = g^{-1}(b)}$

"go backwards"

HOW DO THIS IN MORE VARS?

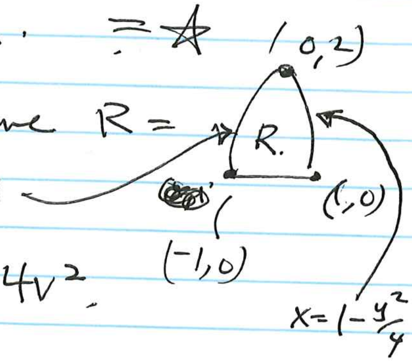
An example will be the best way to see.

BOOK EXAMPLE 1

(But explained differently)

$T = \begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$

Use the change of variables T to set up (without figuring out limits) $\iint_R y\,dA = \star$, where $R =$

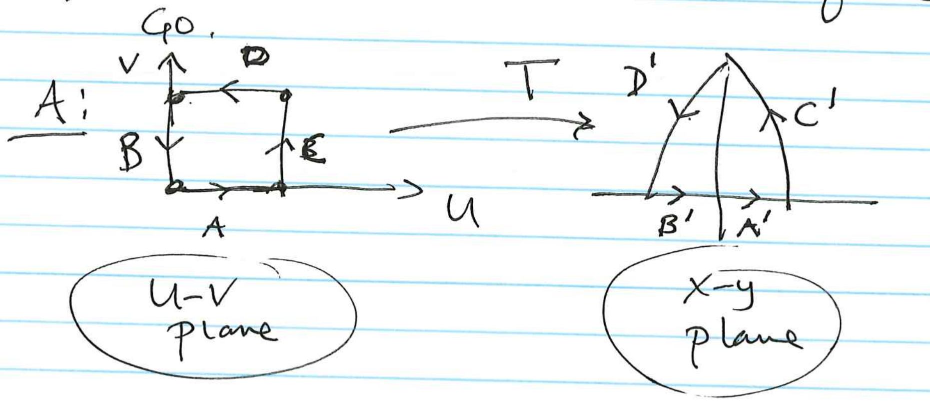


PART 1 $\quad y = 2uv$

PART 2 $\quad \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2.$

So $\star = \iint 2uv \cdot (4u^2 + 4v^2)\,du\,dv.$

DEEP BREATH: Q: WHERE DOES u-v unit square GO.



I'll only do segment A: $v = 0$, $u = 0..1$ so

$y = 2 \cdot u \cdot v = 0$ (since $v = 0$) $\quad \boxed{y = 0}$

$x = u^2 - v^2 = u^2 - 0$

and $u = 0..1$ so $+u^2 = \boxed{0..+1}$ x

Segment A is $y = 0$, $x = 0..+1$, which is Segment A'

Do this out for other segments, and we see

Unit square in u-v plane $\Rightarrow$ R

So the reason for using the transform

$T: \quad x = u^2 - v^2$
$\quad\;\; y = 2uv$ is that the u–v limits are just the unit square.

$\boxed{\text{PART 3}}$

$$\star = \int_{u=0}^{1}\int_{v=0}^{1} 2uv\,(4u^2+4v^2)\,dv\,du.$$

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One more example: 15.9.19.

Use Δ vars $\iint_R xy\,dA$

- [R is bounded by $y=x$, $y=3x$ and ~~parabola~~ hyperbolas $xy=1$, $xy=3$.
- [Δ vars is $x = \frac{u}{v}$, $y=v$.

$$\Rightarrow \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$\Rightarrow$ PARTS 1, 2 give us $\iint \underbrace{\left(\frac{u}{v}\right)}_{x} \cdot \underbrace{(v)}_{y} \cdot \frac{1}{v}\,du\,dv$

$\boxed{\text{KEY}}$ $x\cdot y = 1 \Rightarrow \frac{u}{v}\cdot v = u = 1$!

$x\cdot y = 3 \Rightarrow \frac{u}{v}\cdot v = u = 3$!

$\boxed{u = 1..3}$

$y = x \Rightarrow \frac{u}{v} = v \Rightarrow u = v^2 \Rightarrow v = u^{\frac{1}{2}}$

$y = 3x \Rightarrow \frac{u}{v} = 3v \Rightarrow u = 3v^2 \Rightarrow v = \left(\frac{u}{3}\right)^{\frac{1}{2}}$

$$\int_{u=1}^{3}\int_{v=\left(\frac{u}{3}\right)^{1/2}}^{u^{1/2}} \frac{u}{v}\,dv\,du$$

So $$\int_{u=1}^{3} \int_{v=\left(\frac{u}{3}\right)^{1/2}}^{u^{1/2}} \frac{u}{v}\, dv\, du$$

$$= u\left(\ln u^{1/2} - \ln\left(\frac{u}{3}\right)^{1/2}\right)$$

$$= \frac{1}{2} u \left(\underbrace{\ln u - \ln (u/3)}_{\ln(3)}\right)$$

$$= \int_{u=1}^{3} \frac{\ln(3)}{2} u\, du = \frac{\ln(3)}{2} \Big|_{1}^{3} u^2/2$$

$$= \frac{\ln 3}{2}(4) = \boxed{2 \ln 3}$$

Key is that the Δ vars is often designed to make the limit "nice"

$\boxed{\text{Keep in mind}}$

- THIS IS THE HARDEST SECTION OF CALC III
- YOU CAN MASTER IT BY DOING REPETITIONS