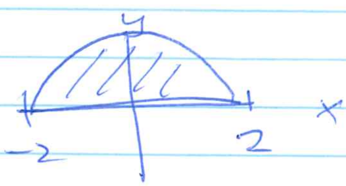
§ 16.1 – 16.2

WARM UP (Review, #48)

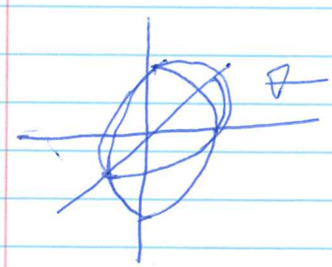
Evaluate $$\int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-y^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$$

$z = \pm\sqrt{4-x^2-y^2} \Rightarrow z^2+x^2+y^2=4$ sphere $r=2$ centered @ origin.

Careful: region in x-y plane is



So: our region in 3D is



← CRAPPY PIX, HALF SPHERE. $r=2$.

Lims in spherical: $\theta = 0..\pi$
$\phi = 0..\pi$ $\quad R$
$\rho = 0..2.$

$$\iiint_R \underbrace{\rho^2 \sin^2\phi \sin^2\theta}_{y^2} \underbrace{\rho}_{\sqrt{x^2+y^2+z^2}} (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_\theta \int_\phi \boxed{\int_\rho \rho^5\, d\rho} \sin^3\phi \sin^2\theta\, d\phi\, d\theta$$

$$\Big|_0^2 \frac{\rho^6}{6} = \frac{32}{3} \int\int_0^\pi \underbrace{\sin^3\phi}_{= (1-\cos^2\phi)\sin\phi} \sin^2\theta\, d\theta$$

$$\Big|_0^\pi -\cos\phi + \frac{\cos^3\phi}{3} = \frac{4}{3} \cdot \frac{32}{3} = \frac{128}{9}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\Rightarrow \frac{128}{9} \int_{\theta=0}^{\pi} \sin^2\theta\, d\theta = \frac{128}{9} \Big|_0^\pi \frac{\theta}{2} - \sin 2\theta = \boxed{\frac{64\pi}{9}}$$

§ 16.1 Vector Fields (VF) Def: A VF on $\mathbb{R}^n$ is a function F which assigns a vector to each point of $\mathbb{R}^n$.

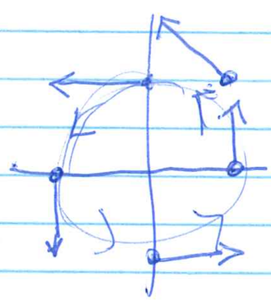
(Usually we "draw" a VF as a bunch of arrows, with tail at point of origin)

Example 1

$$\vec{F}(x,y) = -y\,i + x\,j$$

Plot some points

P	$\vec{v}$
1 0	j
0 1	$-i$
-1 0	$-j$
0 -1	i



looks like rotation – check at (1,1)

$$(1,1) \to -i + j$$

Typical VF – fluid flow, velocity, gravity, E+M.

gravity $\vec{F} = \dfrac{-mMG}{|x|^3}x$ (m, M masses, G grav. const)

$$= \frac{-mMG}{|x|^2} \cdot \frac{x}{|x|} \text{ unit vector}$$

Newton

So $|\vec{F}| = \dfrac{-mMG}{|x|^2}$ $\quad \bar{x} = (x,y,z)$, $|x| = \sqrt{x^2+y^2+z^2}$

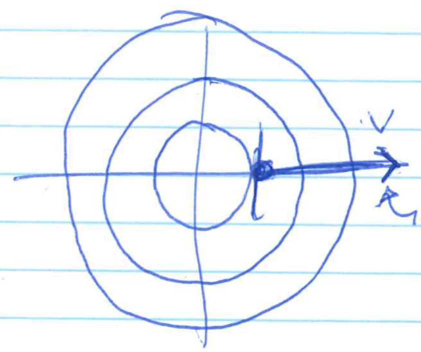
$$= \frac{-mMG}{r^2} \quad \text{"inverse square" law}$$

The most important VF: gradient field

recall if f is a function, $\nabla f = f_x i + f_y j + \cdots$

★ [Key prop of ∇f: $\perp$ to level curves of f.]

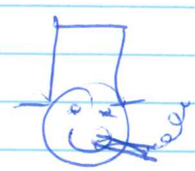
eg $f(x,y) = x^2+y^2$



$\perp$ to level curve $x^2+y^2=1$

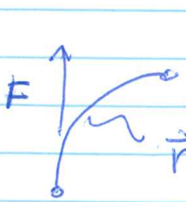
$\nabla f = 2xi + 2yj$

at $(1,0) = 2i$ $\vec{v}$

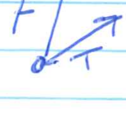


Def: A VF is conservative if $\vec{F} = \nabla f$; f is a potential fn for $\vec{F}$.

Who cares? How do we compute work done by a VF as it moves a particle from a to b? along a curve C

A: 

$\vec{r}(t)$ = path $\frac{\vec{r}'(t)}{|\vec{r}'(t)|} = T$

we want 

"amount" of $\vec{F}$ pushing in direction T

$= \vec{F} \cdot T$

Then we'll sum up all the contributions

$$\star\star \int_C \vec{F} \cdot T \cdot \underset{\sim}{ds}$$

little piece of the arc C

<u>Key</u> A kind of Fundamental Thm of Calc.

if $F = \nabla f$, $\star\star$ is indep of C

"path indep"

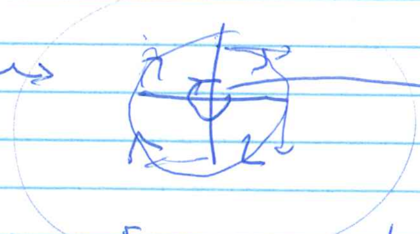
<u>Who cares, II</u>

$$\underset{\text{Nasty path}}{\int F \cdot T\, ds} = \int F \cdot T\, ds$$

easy path

☹ if $F = \nabla f$ ☺

<u>Class</u> (6) Sketch $\vec{F}(x,y) = \dfrac{y\,i - x\,j}{\sqrt{x^2+y^2}}$ $\rightsquigarrow$

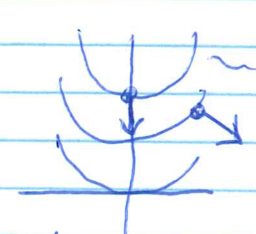


strong near origin

weaker far out

(21) Find gradient field + sketch: $f = x^2 - y$

$\nabla f = (2x\,i - j)$



level curves $y = x^2 + c$

$\nabla f \perp$ level curves.

Q: is $xy\,i + y\,j$ conservative?

does there exist $f(x,y)$, $f_x = xy$, $f_y = y$?

$f_x = xy \Rightarrow f = \dfrac{x^2}{2} y + C$ but $C = g(y)$!

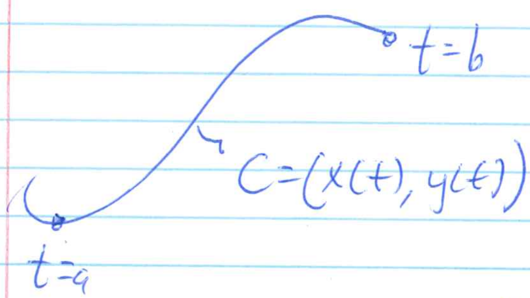
$f = \dfrac{x^2}{2} y + g(y) \Rightarrow f_y = \dfrac{x^2}{2} + g'(y) \overset{?}{=} y$ No! $g'(y) = y - \dfrac{x^2}{2}$

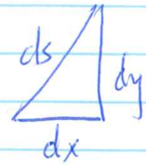
but $g(y)$ was fn only of y.

§16.2

Integrating functions / VF along a curve.

Recall when we did arclength.





$$ds = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$

$$\sum ds = \int_{t=a}^{b} \sqrt{(x')^2 + (y')^2}\, dt.$$

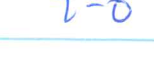
Now, suppose we want to integrate $f(x,y)$ along C

$$\Rightarrow \sum f(x(t), y(t))\, ds \Rightarrow \int_{t=a}^{b} \underbrace{f(x(t), y(t))}_{\text{just plug in param. to } f}\, ds$$

Ex 1 $\displaystyle\int_{C} (2 + x^2 y)\, ds$

C = top 1/2 unit circle

Soln:



$x = \cos t$, $y = \sin t$, $t = 0 .. \pi$. $ds = \sqrt{(-\sin)^2 + (\cos^2)}\, dt = dt.$

$$\Rightarrow \int_{t=0}^{\pi} \Big(2 + \underbrace{\cos^2 t \sin t}_{f(\cos t, \sin t)}\Big)\, dt = \Big[\, 2t - \frac{\cos^3 t}{3} \Big]_{t=0}^{\pi} \boxed{= 2\pi + 2/3}$$

Class: $\int (x^2 y^3 - \sqrt{x})\, dy$

along $y = \sqrt{x}$, from $(1,1)$ to $(4,2)$.

Soln: Think $x = t$, $y = t^{1/2}$. $x = t$ goes $1..4$

$dy = \frac{1}{2} t^{-1/2}\, dt$

$$\int_{t=1}^{4} (t^{7/2} - t^{1/2}) \tfrac{1}{2} t^{-1/2} dt$$

$$= \frac{1}{2}\int_1^4 (t^3 - 1)\, dt = \frac{1}{2} \Big|_1^4 \frac{t^4}{4} - t = \boxed{\frac{243}{8}}$$

$\boxed{\int_{\text{CURVE}} \text{VF}}$

As noted earlier, we want to interpret <u>work</u> $\vec{F}$ does moving a particle along C.

$$= \int_C F \cdot \vec{T}\, ds, \quad T = \frac{r'(t)}{|r'(t)|}.$$

but what is ds? if $r = (x(t), y(t))$

$r' = (x', y')$

$|r'| = \sqrt{(x')^2 + (y')^2} = ds$!

So $\int_C F \cdot T\, ds = \int F \cdot \frac{r'(t)}{|r'(t)|} |r'(t)|\, dt$ CANCEL !

$$= \int \vec{F} \cdot r'(t)\, dt,$$

Me: ~~$\vec{F} = xy\, i + 2y^2 j$, $\vec{r}(t) = \ldots$~~ 19 $\vec{F} = xy^2 i - x^2 j$
$\vec{r} = t^3 i + t^2 j$, $t = 0..1$

Soln: $\int_{t=0}^{1} (t^7, -t^6) \cdot (3t^2, 2t)\, dt = \int_{t=0}^{1} 3t^9 - 2t^7\, dt$ ☺

Class: 20 $\vec{F} = (x + y^2, xz, y+z)$, $r = (t^2, t^3, -2t)$ $t = 0..2$.

$$\int \underbrace{(t^2 + t^6, -2t^3, t^3 - 2t)}_{\vec{F}(r(t))} \cdot \underbrace{(2t, 3t^2, -2)}_{\vec{r}\,'(t)}\, dt$$

easy! ☺