§ 16.3 ⓪

Warm up 1 $\vec{F}(x,y) = x^2 i + xy\, j$. Particle moves around circle $x^2+y^2=4$ counterclockwise $0..2\pi$.

a) find work done

b) Plot VF and explain answer.

Soln a) • parameterize the circle

$$\vec{r}(t) = (2\cos t, 2\sin t) \quad t = 0..2\pi$$

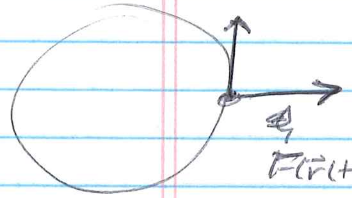
• $\text{Work} = \int_0^{2\pi} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$r'(t) = \langle -2\sin t, 2\cos t \rangle$

$$= \int_0^{2\pi} \langle (2\cos t)^2, 4\cos t \sin t \rangle \langle -2\sin t, 2\cos t \rangle$$

$$= 8\int_0^{2\pi} \underbrace{(\cos^2 \sin - \cos^2 \sin)}_{0}\, dt = \boxed{0}$$

b)



$F(\vec{r}(t))$

$F(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$

at $t=0$ $F(\vec{r}(t)) = \langle 2, 0 \rangle$

$\vec{r}\,'(t) = \langle 0, 1 \rangle$

So F does not contribute!

WARM UP 2 $\int_C x^2 y\, ds$, $C = \langle \cos t, \sin t, t \rangle$ $0 \le t \le \frac{\pi}{2}$.

$ds = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}\, dt$

$$= \int_{t=0}^{\pi/2} \underbrace{\cos^2 t}_{x^2} \cdot \underbrace{\sin t}_{y} \underbrace{\sqrt{2}\, dt}_{ds} = \sqrt{2} \Big|_0^{\pi/2} -\frac{\cos^3}{3} = \boxed{\frac{\sqrt{2}}{3}}$$

WARM UP 3 Is $\langle 2xy, x^2-3y^2 \rangle$ conservative?

$$\begin{array}{cc} \| ? & \| ? \\ f_x & f_y \end{array}$$

$$2xy = f_x \Rightarrow \underbrace{\int 2xy\,dx}_{= x^2y} = \int f_x\,dx = f(x,y) + g(y)$$

$\boxed{\text{Candidate}}$

$$\Rightarrow \tilde{f}(x,y) = x^2y + g(y), \quad \frac{\partial}{\partial y}\tilde{f} = x^2 + g'(y) \overset{?}{=} x^2 - 3y^2$$

OK if $g'(y) = -3y^2$

$\Rightarrow g(y) = -y^3 + C.$

$\boxed{\text{CHECK}}$ $f(x,y) = x^2y - y^3 + C$

$f_x = 2xy$, $f_y = x^2 - 3y^2$ ✓

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§16.3 Why do we care if a VF is conservative?

A: Theorem $\left[\begin{array}{l} \text{if } F = \nabla f, \\ \text{and } C = r(t) \\ \quad t = a..b \end{array}\right]$ then $\displaystyle\int_C \vec{F} \cdot d\vec{r} = f(r(b)) - f(r(a))$

((assuming C is smooth (piecewise) and ∇f continuous on C))

Proof: (in $\mathbb{R}^3$)

$$\vec{\nabla} f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\vec{r}\,'(t) = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

$$\vec{\nabla} f \cdot d\vec{r} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \frac{d}{dt}\left(f(r(t))\right)$$ ⟸ JUST FUND. THM OF CALC

$$\int_{x=a}^{b} \frac{df}{dx} = f(b) - f(a)$$

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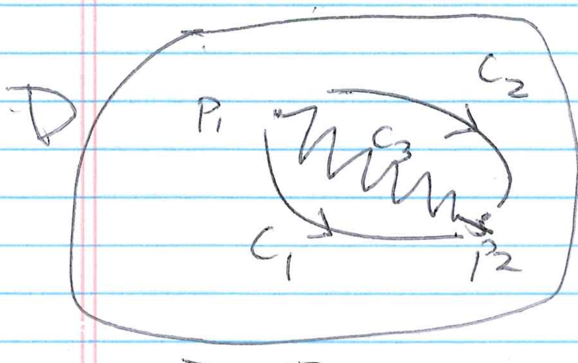
Def: $\int_C \vec{F} \cdot d\vec{r}$ is path indep (PID)

if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2 = \vec{r}_2(t)} F \cdot d\vec{r}$

C_1: $\vec{r}_1(t)$

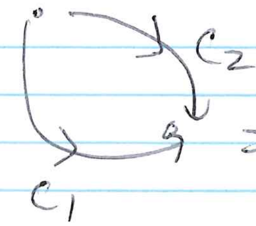
for any paths in D w/ same start/finish.



By Thm, a conservative VF is PID.

NOTICE $\int_C \vec{F} \cdot d\vec{r}$ is PID ⟺ $\oint_C \vec{F} \cdot d\vec{r} = 0$

for all C ← closed curve = start = finish

Proof is obvious:  $\Rightarrow \underbrace{\int_{C_1} - \int_{C_2}}_{C_1 + -C_2 \,=\, \text{closed path}} = 0$

$-C_2$ = traverse C_2 backwards

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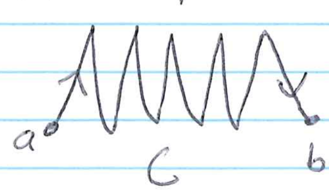
Surprise: D open connected region, $\vec{F}$ cont

Then $\int_C F \cdot dr$ PID $\Rightarrow$ Conservative

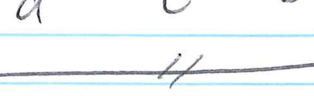
$\boxed{\text{PUNCHLINE}}$ Conservative VF are exactly those VF where it does not matter how we go from a to b.

So, suppose you have to do



$\int_C F \cdot dr$ ( ). If F is conservative, then can just do

$\int_{C'} \vec{F} \cdot d\vec{r}$ ( )



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So, conservative VF are nice.

How TELL?

In $\mathbb{R}^2$ Suppose $F = P(x,y)\,i + Q(x,y)\,j$

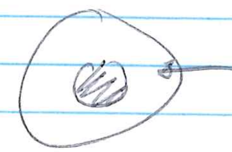
If $\vec{F} = \nabla f$, then $P = f_x$, $Q = f_y$

Now, mixed partials agree, so $f_{xy} = f_{yx}$ (if they exist and are cont)

$P_y = f_{xy} = f_{yx} = Q_x$

$\boxed{\text{Necessary}}$ condition.

If D has no holes, this is also sufficient



BAD D

FUN APPLICATION

F moves object on path $r(t)$

$r(a) = A$, $r(b) = B$. Find work done.

Newton $F = ma = m\vec{r}''(t)$

$$W = \int_C F \cdot dr = \int \underbrace{F(\vec{r}(t))}_{m r''(t)} \cdot \vec{r}'(t)$$

$$= m \cdot \frac{1}{2} \int \underbrace{2 r''(t) \cdot r'(t)}_{\frac{d}{dt}(r' \cdot r')}$$

FTC

$$= \frac{m}{2} |r'(t)|^2 \Big|_a^b = \frac{m}{2}\left(|r'(b)|^2 - |r'(a)|^2\right)$$

$=$ Kinetic energy @ finish $-$ Kinetic energy @ Start.

Kinetic Energy is $\frac{m}{2} \cdot \underbrace{|V(t)|}_{\text{speed}}{}^2$ (velocity $= \vec{V}$)

REPS TIME!

WHICH ARE CONSERVATIVE

- $\langle 3x^2-2y^2,\ 4xy+3 \rangle$ $\qquad -4y \neq 4y$ NO
- $\langle ye^x+\sin y,\ e^x+x\cos y \rangle$ $\qquad e^x+\cos y = e^x+\cos y$ YES

Reminder: If F is conservative, to find the potential f $(\vec{F}=\nabla f)$ we backsolve

$F = f_x\, i + f_y\, j + f_z\, k$

① integrate f_x wrt x, add $g(y,z)$ as const

② Do for remaining partials.

- $\langle 2xz+y^2,\ 2xy,\ x^2+3z^2 \rangle$ Is it conserv., if so, find potential

$$\begin{array}{ccc} 2xz+y^2 & 2xy & x^2+3z^2 \\ \overset{?}{\|} & \overset{?}{\|} & \| \\ f_x & f_y & f_z \end{array}$$

$\frac{\partial f_x}{\partial y} \overset{?}{=} \frac{\partial f_y}{\partial x}$ (Do for other z)

$2y = 2y$ ✓

Find f:

$$\int 2xz+y^2\, dx = x^2z+xy^2+g(y,z)$$

$\Downarrow$ hit w/ $\partial/\partial y$

$2xy+\frac{\partial g}{\partial y} \overset{?}{=} 2xy$

$\Rightarrow \frac{\partial g}{\partial y}=0 \Rightarrow g=g(z)$

$x^2z+xy^2+g(z)$

$\Downarrow \partial/\partial z$

$x^2+g'(z) \overset{?}{=} x^2+3z^2$

$\Rightarrow g'(z)=3z^2 \Rightarrow g(z)=z^3+C$

$\Rightarrow f = x^2z+xy^2+z^3+C$ (CHECK IT).