§ 16.4.

CAT 1

Find work done in moving from $(0,1)$ to $(2,0)$. $\vec{F} = (e^{-y}, -xe^{-y}) = \langle P, Q \rangle$

Soln: We weren't given a path, so can only solve if $\vec{F} = \nabla f$.

Check: $P_y = Q_x$?

$$\frac{\partial}{\partial y}\big(\underset{P}{e^{-y}}\big) = -e^{-y} \cdot \frac{\partial}{\partial x}\big(\underset{Q}{-xe^{-y}}\big) = -e^{-y} \checkmark$$

Conservative $\Rightarrow$ PID, find potential fn f

$$\int \underset{\underset{P}{\uparrow}}{e^{-y}}\, dx = xe^{-y} + g(y) \xrightarrow{\frac{\partial}{\partial y}} -xe^{-y} + g'(y) \overset{?}{=} Q$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

$f = xe^{-y} + C$. **Check:** $\frac{\partial f}{\partial x} = P$ ✓

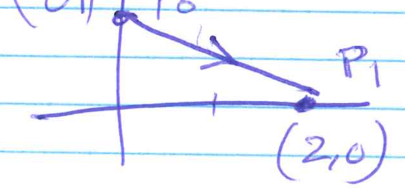
$\frac{\partial f}{\partial y} = Q$ ✓

Now, our theorem says $\int_{t=a}^{b} \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))$

$$= xe^{-y} + c \Big|_{01}^{20} = (2+c) - (0+c)$$

$$\boxed{=2}$$

Suppose we had a panic attack, and decided we had to parameterize a path (note; this is a problem, because $\int_C F\cdot d\vec{r}$ depends on C, unless $F=\nabla f$) but drive on.



taking $(1-t)p_0 + t(p_1)$ $[t=0..1]$

gives $(1-t)(0,1)+t(2,0) = (2t,1-t)$

check : @ $t=0$, $(0,1)$ ✓
$t=1$ $(2,0)$ ✓

$$r'(t) = \langle 2,-1\rangle dt$$

$$F(r(t)) = \langle e^{-(\underset{\underset{y}{\uparrow}}{1-t})}, -\overset{\overset{x}{\downarrow}}{2t} e^{-(1-t)}\rangle \langle 2,-1\rangle dt$$

$$= \int_{t=0}^{1} \underset{\underset{\text{easy}}{\uparrow}}{2e^{t-1}} + \underset{u}{\underbrace{2t}}\underset{\underset{dv}{\uparrow}}{\underbrace{e^{t-1}}}\,dt$$

integration by parts.

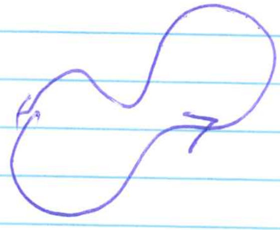
$$\int u\,dv = uv - \int v\,du$$

DO THIS! YOU'LL STILL GET $\boxed{2}$

§ 16.4 Greens Thm (Super Important)

Let C be a $\underset{\substack{\uparrow \\ \text{doesnt} \\ \text{cross} \\ \text{itself}}}{\text{simple}}$ $\underset{\substack{\uparrow \\ \text{start} \\ \text{pt} \\ = \\ \text{end} \\ \text{pt} \\ \text{(a track)}}}{\text{closed}}$ curve

oriented counterclockwise e.g.



THEN $$\int_C P\,dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

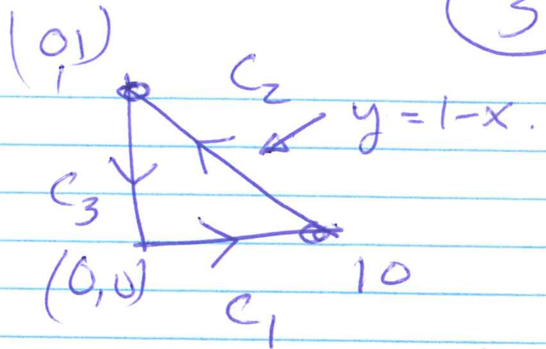
Asymmetry in formula! sucks.

So: do $P\,dx + Q\,dy$ (in order: $\begin{matrix} P < Q \\ x < y \end{matrix}$)

then Swap $\left(\underset{\substack{\uparrow \\ \text{single} \int}}{1} < \underset{\substack{\uparrow \\ \iint}}{2} \right)$ $\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$

Q: $\boxed{\text{WHO CARES?}}$

Ⓐ $$\int_C \overset{\overset{P}{\downarrow}}{x^4}\,dx + \overset{\overset{Q}{\downarrow}}{xy}\,dy$$



Can do directly as $\int_{C_1} + \int_{C_2} + \int_{C_3}$ ☆

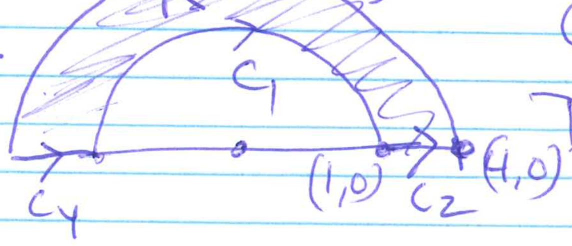
By Greens Thm, it is $\iint_{◣} \underset{Q_x}{\underset{\uparrow}{y}} - \underset{P_y}{\underset{\uparrow}{0}}\; dx\,dy$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} y\,dy\,dx = \int \left.\frac{y^2}{2}\right|_0^{1-x} = \frac{1}{2}\int_{x=0}^{1}(1-x)^2\,dx = -\frac{1}{6}\left.(1-x)^3\right|_0^1 \boxed{= 1/6}$$

easy peasy!

$\boxed{\text{At Home}}$ -> do ☆.

Ⓑ Class



$C = C_1 + C_2 + C_3 + C_4$

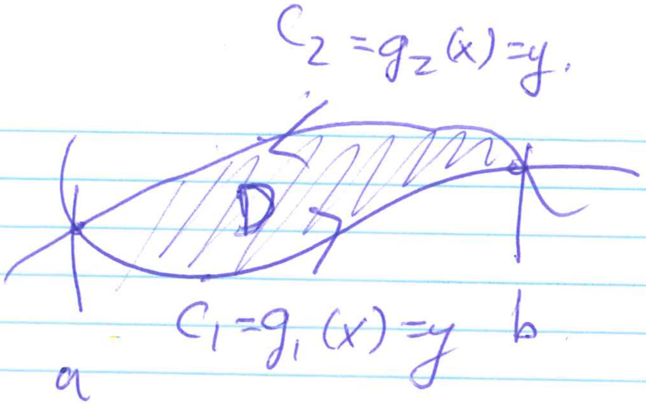
Do $\int_C \underset{P}{y^2}\,dx + \underset{Q}{3xy}\,dy$.

☹

→ either parameterize 4 arcs. OR easy ☺

$$\iint_R \underset{Q_x}{\underset{\uparrow}{3y}} - \underset{P_y}{\underset{\uparrow}{2y}}\; dy\,dx = \iint_R y\,dx\,dy = \int_{\theta=0}^{\pi}\int_{r=1}^{2} r\sin\theta\, r\,dr\,d\theta = \boxed{\frac{14}{3}}$$

Proof



$$\iint_D \frac{\partial P}{\partial y}\,dA = \int_{x=a}^{b}\int_{y=g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$$

$$\overset{(FTC)}{=} \int_{x=a}^{b} P(x,g_2(x)) - P(x,g_1(x))\,dx \quad \circledast$$

OTOH, $\int_C P\,dx = \int_{C_1} P(x,g_1(x))\,dx + \int_{C_2} P(x,g_2(x))\,dx$

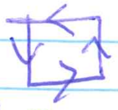
$$= \int_a^b P(x,g_1(x)) \overset{C_2\,\downarrow}{-} \int_a^b P(x,g_2(x)$$

$$= -\int_a^b P(x,g_2) - P(x,g_1) = -\circledast$$

$$\int P\,dx \longrightarrow -\iint \frac{\partial P}{\partial y}\,dA$$

(BUT) $\int Q\,dy \longrightarrow \iint \frac{\partial Q}{\partial x}\,dA$ ($\boxed{\text{orientation}}$ is a choice we made.

$\boxed{\text{RMK}}$ WORKS IF • piecewise sm

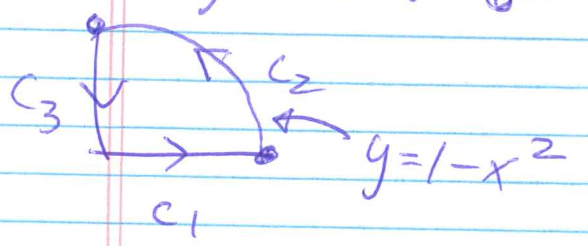


• WITH HOLES (Region on Left

⑤

$$\boxed{\text{Rep5}}$$

$$\int \overset{P}{x}\,dx + \overset{Q}{y}\,dy$$



Directly $$\int_{t=0}^{1} t\,dt + \int_{t=1}^{0} \left(t+(-2t)(1-t^2)\right)dt + \int_{t=1}^{0} t\,dt$$

$x=t$, $y=0$ } C_1

$x=t$, $y=1-t^2$ C_2

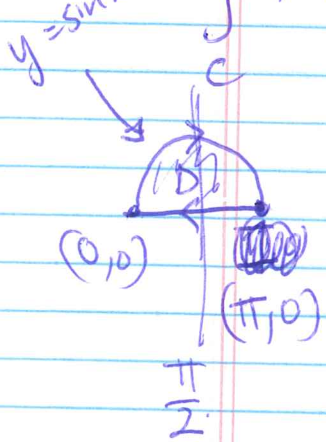
$t=t$, $x=0$ C_3.

$$\boxed{\tfrac{1}{2}} \qquad \boxed{0} \qquad \boxed{-\tfrac{1}{2}}$$

OTOH: (Green) $$\iint \overset{Q_x}{0} - \overset{P_y}{0} = \iint 0 = 0 \checkmark$$

$$\int_C F\cdot dr \qquad \vec{F} = \langle \overset{P}{\sqrt{x}+y^3},\ \overset{Q}{x^2+\sqrt{y}} \rangle$$

$$dr = \langle dx, dy \rangle \implies \int P\,dx + Q\,dy$$



Green

$$\iint_D (2x-3y^2)\,dy\,dx = \left. 2xy-y^3 \right|_0^{\sin x} \int_{x=0}^{\pi} 2x\sin x - \underbrace{\sin^3 x}_{(1-c^2)\sin}\,dx$$

$$\int_{X\,0}^{\pi}\int_{y=0}^{\sin x}$$

$\uparrow$ $\int u\,dv$.

Grind it out

$$\int_0^{\pi} (1-c^2)\ \sin\ dx = \Big|_0^{\pi} -\cos + \frac{c^3}{3} = \overset{top}{\left(1-\frac{1}{3}\right)} - \overset{bot}{\left(-1-\frac{1}{3}\right)}$$

$$= 4/3.$$

$$\int_0^{\pi} \underbrace{x}_{u}\ \underbrace{\sin x}_{dv}\ dx \qquad \int u\,dv = uv - \int v\,du$$

$v = -\cos x$

$du = dx$

$$= -x\cos x - \int -\cos x\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x\ \Big|_0^{\pi}$$

$$= \overset{top}{(\pi)} - \overset{bot}{0}$$

$$\boxed{= \pi + \frac{4}{3}}$$