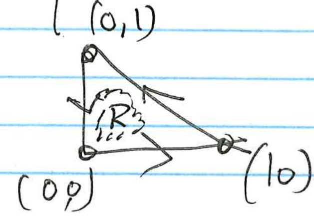
§ 16.5

WARM UP

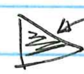
- Find work done by $\vec{F}(x,y) = \langle x(x+y), xy^2 \rangle$ to move particle from $(0,0)$ on path



R = area enclosed by C

- Soln: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy \overset{\text{GREEN}}{=} \iint_R Q_x - P_y \, dA$

$$Q_x = y^2,\ P_y = x \Rightarrow \iint (y^2 - x)\, dA$$



$x + y = 1 \Rightarrow y = 1 - x$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} (y^2 - x)\, dy\, dx = \int_{x=0}^{1} \left. \frac{y^3}{3} - xy \right|_0^{1-x} = \int_{x=0}^{1} \frac{(1-x)^3}{3} - x(1-x)\, dx$$

$$= \left. -\frac{(1-x)^4}{12} - \frac{x^2}{2} + \frac{x^3}{3} \right|_0^1 = \boxed{\frac{1}{12}}$$

TODAY § 16.5 $\boxed{\text{Curl}}$ and $\boxed{\text{Divergence}}$

(VF = vector field)

Curl : VF → VF

Div : VF → functions

Def $\quad \text{Curl}\left(f(x,y,z)\,i + g(x,y,z)\,j + h(x,y,z)\,k\right) = \langle f, g, h\rangle$

"Symbolic" Determinant

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix}$$

(Can't really ("officially") put i,j,k inside a determinant)

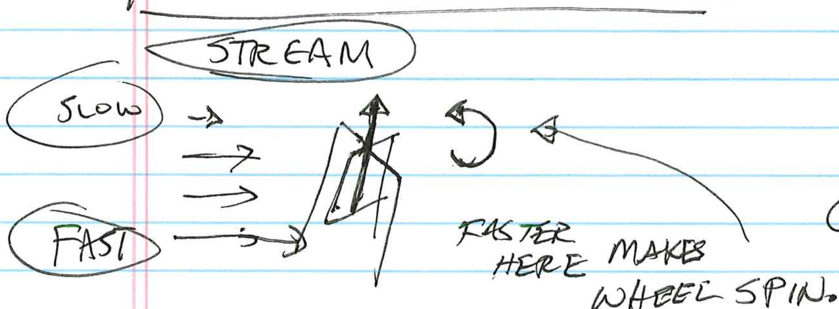
$$= i \begin{vmatrix} \partial_y & \partial_z \\ g & h \end{vmatrix} - j \begin{vmatrix} \partial_x & \partial_z \\ f & h \end{vmatrix} + k \begin{vmatrix} \partial_x & \partial_y \\ f & g \end{vmatrix}$$

Example $\quad \text{Curl}\langle yz, xz, xy\rangle$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix} = i\,(z-z) - j\,(y-y) + k\,(z-z) = 0$$

WHAT IT MEANS

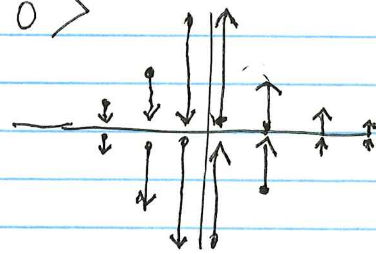
DROP A LITTLE PADDLE WHEEL INTO A V.F.



IF WHEEL SPINS, CURL = AXIS OF ROTATION.

Example: Consider the VF

$\langle 0, \frac{1}{x}, 0 \rangle$

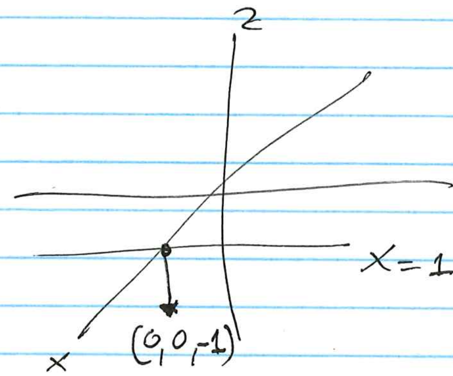


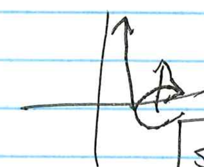
(just x-y picture)

CURL: $\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & \frac{1}{x} & 0 \end{vmatrix} = 0i - 0j - \frac{1}{x^2}k$

Say at $x = 1$
curl $= \langle 0, 0, -1 \rangle$

$\boxed{\text{CURL} = 0 \Leftrightarrow \text{IRROTATIONAL}}$





spin axis in z-direction

$\boxed{\text{DIVERGENCE}}$

$\vec{F} = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$

$\nabla F = \operatorname{div} \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

(also, works in $\mathbb{R}^n$, whereas Curl is $\mathbb{R}^3$)

If we think of f as flow in x-direction (i component)
g as flow in y direction, h as flow in z-dir,
then ∇F = net flow in/out at a point.

$\boxed{\text{DIV} = 0 \Longleftrightarrow \text{INCOMPRESSIBLE}}$

$\boxed{\text{EX}}$ $\nabla(\langle -y(2+x), x, yz\rangle)$

$$= \underset{\downarrow}{\partial/\partial x} + \underset{\downarrow}{\partial/\partial y} + \underset{\downarrow}{\partial/\partial z}$$

hit with / hit with / hit with

$$-y + 0 + y = 0$$

$\boxed{\text{Incompressible}}$

$\boxed{\text{EX}}$ Find curl $\langle xz, xyz, -y^2\rangle$, compute its divergence

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xz & xyz & -y^2 \end{vmatrix} = \langle -2y - xy, x, yz\rangle)$$

we already computed div!

In this example, div(curl $(\vec{F})) = 0$

Suppose: on a test you had to prove

Thm div (curl $(\vec{F})) = 0$ for all VF $\vec{F}$

Proof: $F = \langle f, g, h\rangle$. curl $F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{vmatrix} = \langle h_y - g_z, f_z - h_x, g_x - f_y\rangle$

div $(h_y - g_z, f_z - h_x, g_x - f_y)$

$$= \frac{\partial}{\partial x}(h_y - g_z) + \frac{\partial}{\partial y}(f_z - h_x) + \frac{\partial}{\partial z}(g_x - f_y) = \left\{\begin{array}{l} h_{yx} - g_{zx} \\ f_{zy} - h_{xy} \\ g_{xz} - f_{yz} \end{array}\right. =$$

DONE

Show $\int_C \vec{F}(x,y)\cdot d\vec{r} = \iint_D \text{curl}F \cdot \vec{k}\, dA.$ ☆

$\vec{F}(x,y) = P(x,y)\,i + Q(x,y)\,j + 0\,k$ (in $\mathbb{R}^2$, no z part)

So $\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = 0i + 0j + (Q_x - P_y)k$

and $(Q_x - P_y)\vec{k}\cdot\vec{k} = Q_x - P_y \Rightarrow$ We get Green!

☆ is Greens Thm, which is a special case of Stokes Thm (16.8)

Reps (7.) compute curl + divergence

$$\langle \underset{f}{e^x \sin y},\ \underset{g}{e^y \sin z},\ \underset{h}{e^z \sin x} \rangle$$

CURL $\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ e^x \sin y & e^y \sin z & e^z \sin x \end{vmatrix} = \langle 0 - e^y \cos z,\ -(e^z \cos x - 0),\ 0 - e^x \cos y \rangle$

DIV $\frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g + \frac{\partial}{\partial z} h = e^x \sin y + e^y \sin z + e^z \sin x = \vec{F}$

$\boxed{\text{Reps}}$

(assumption: second partials cont)

- Prove $\text{curl}(\nabla f) = 0$. Use this to test if $\langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle = \vec{F}$ is conservative. If so, find potential fn.

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y^2z^3, & 2xyz^3, & 3xy^2z^2 \end{vmatrix} = \langle \overset{0}{6xyz^2 - 6xyz^2}, \overset{0}{3y^2z^2 - 3y^2z^2}, \overset{0}{2yz^3 - 2yz^3} \rangle$$

$\underline{\text{CONSERVATIVE}}$

(THIS IS JUST AN ENCODING OF OUR PREV. TEST)

- $\frac{\partial f}{\partial x} = y^2z^3 \Rightarrow \tilde{f} = xy^2z^3 + g(y,z)$

$$\downarrow$$

$$\partial/\partial y = 2xyz^3 + \frac{\partial g}{\partial y} \overset{?}{=} 2xyz^3 \Rightarrow \frac{\partial g}{\partial y} = 0$$

$$\tilde{f} = xy^2z^3 + g(z)$$

$$\downarrow \partial/\partial z$$

$$3xy^2z^2 + \partial g/\partial z \overset{?}{=} 3xy^2z^2 \Rightarrow \frac{\partial g}{\partial z} = 0$$

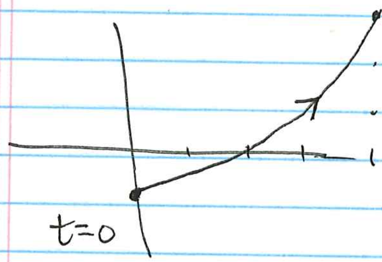
$$\Rightarrow \boxed{f = xy^2z^3 + C}$$

CHECK by taking $\partial_x, \partial_y, \partial_z$, compare to $\vec{F}$.

BONUS REP TEST PRACTICE PROBLEM

Find work done by $\vec{F} = \langle x+y,\ xy \rangle$ to move a particle along path

$\vec{r}(t) = \langle 2t,\ t^2-1 \rangle \quad t = [0..2]$.



SOLN

$\vec{r}\,'(t) = \langle 2,\ 2t \rangle$

$\vec{F}(\vec{r}(t)) = \langle \overset{x+y}{t^2+2t-1},\ \overset{xy}{2t^3-2t} \rangle$

$$= \int_{t=0}^{2} \underbrace{\langle t^2+2t-1,\ 2t^3-2t \rangle \langle 2,\ 2t \rangle}\, dt$$

$$\int_{t=0}^{2} \underbrace{2t^2+4t-2+4t^4-4t^2}\, dt$$

$$\int_0^2 4t^4-2t^2+4t-2\, dt$$

$$= \Big|_0^2 \frac{4t^5}{5} - \frac{2t^3}{3} + 2t^2 - 2t$$

$$\boxed{= 4\left(\frac{2^5}{5}\right) - \frac{2^4}{3} + 8 - 4}$$