§ 16.6

Warm up!

19) Is there a VF s.t. $\text{curl}(\vec{F}) = \langle x \sin y, \cos y, z - xy \rangle$ ☆

A: we know $\text{div}(\text{curl}(F)) = 0$.

So if $\text{div}(☆) \neq 0$, it can't be curl

$\text{div} = \sin y - \sin y + 1 = 1 \neq 0$ $\boxed{\text{NO}}$

6) Compute div, curl for

$\langle \ln(2y+3z), \ln(x+3z), \ln(x+2y) \rangle$

div! $= 0 \Longrightarrow \partial/\partial x = 0 + \partial/\partial y = 0 + \partial/\partial z = 0$

curl

$$\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ \ln(2y+3z) & \ln(x+3z) & \ln(x+2y) \end{vmatrix}$$

$$= \left\langle \frac{2}{x+2y} - \frac{3}{x+3z}, -\left(\frac{1}{x+2y} - \frac{3}{2y+3z}\right), \frac{1}{x+3z} - \frac{2}{2y+3z} \right\rangle$$

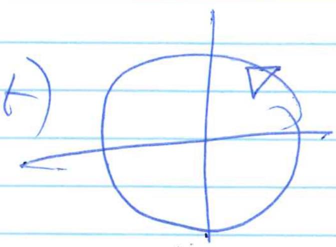
§ 16.6

Parametric surfaces

$x = f(u,v), \quad y = g(u,v), \quad z = h(u,v)$

like parametric curve

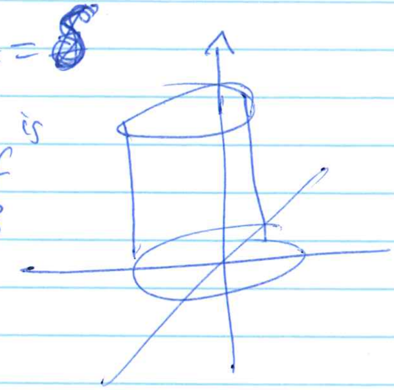
$(x = \cos t, \ y = \sin t)$



but 3 vars, 2 params $\Rightarrow$ surface

Example: $x = 3\cos t, \ y = 3\sin t, \ z = s$

params are t, s. surface is cylinder of radius 3



Example sphere of radius 1

$x = 1 \sin\phi \cos\theta$

$y = 1 \sin\phi \sin\theta$

$z = \cos\phi$

$(\rho = 1)$

x, y, z functions of 2 parameters.

Example 7 find parametric eqns for surface $z = 2\sqrt{x^2+y^2}$ (positive $\sqrt{\ }$)

Soln:
$$\left.\begin{array}{l} x = x \\ y = y \\ z = 2\sqrt{x^2+y^2} \end{array}\right\} \text{ super easy!}$$

Analogous to: find parametric rep of curve
$$\left.\begin{array}{l} y = x^2+1 \\ x = x \end{array}\right\}$$

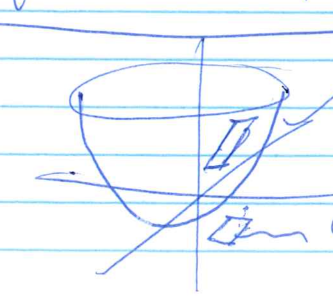
Main Technical "point" today

(1) how do you find normal to tangent plane to para. surf?

(2) How do you find surface area of a parametric surface?

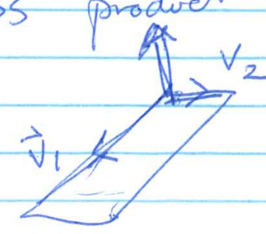
FLASHBACK:

Find surface area for $z = x^2+y^2$ above the unit disk

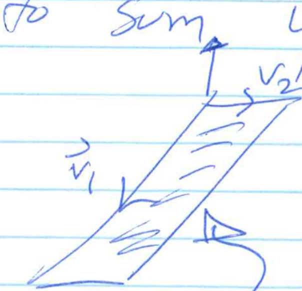


$$\left.\begin{array}{l} \langle dx, 0, f_x\, dx\rangle \ \vec{v}_1 \\ \langle 0, dy, f_y\, dy\rangle \ \vec{v}_2 \end{array}\right\} \text{ cross product } v_1 \times v_2$$

dx dy downstairs



we want to sum up the little // ograms



we know area $= |v_1 \times v_2|$.

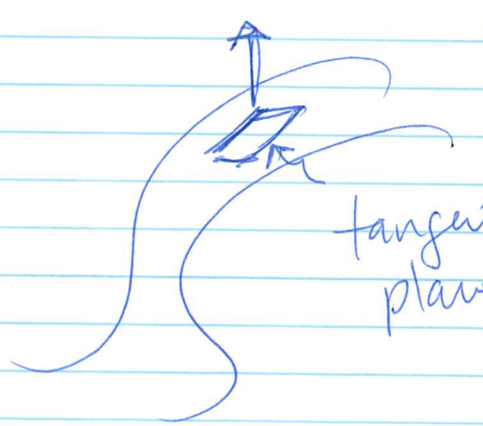
so we did
$$\begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

and get Area of // ogram $= |\langle -f_x, -f_y, 1 \rangle|$

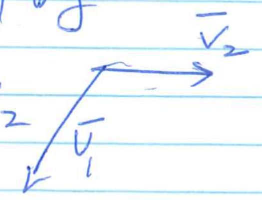
$$= \sqrt{f_x^2 + f_y^2 + 1}$$

Then $SA = \iint_{\text{Region in } x\text{-}y \text{ plane}} \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$

__Notice__ The __normal__ to the surface is exactly $\vec{V}_1 \times \vec{V}_2$



tangent plane spanned by $\bar{V}_1, \bar{V}_2$



What's the trick? we do the same thing!

$r(s,t) = x = f(s,t),\ y = g(s,t),\ z = h(s,t)$

Normal to tangent plane is

$$\vec{r}_s \times \vec{r}_t$$

and area of little surface patch is $|r_s \times r_t|$.

Example:

40 Find SA of part of plane

$$r(u,v) = \langle u+v,\ 2-3u,\ 1+u-v \rangle$$

above $0 \le u \le 2$, $-1 \le v \le 1$.

Soln: $r_u = \langle 1, -3, 1 \rangle$

$r_v = \langle 1, 0, -1 \rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & -3 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \langle 2, 2, 3 \rangle$$

$$|r_u \times r_v| = \sqrt{2^2+2^2+3^2} = \sqrt{17}$$

$$\int_{u=0}^{2}\int_{v=-1}^{1} \sqrt{17}\, dv\, du = \boxed{\sqrt{17} \cdot 4}$$

CLASS REPS

48 $\bar{r}(u,v) = \langle u\cos v,\ u\sin v,\ v \rangle$

$0 \le u \le 1,\ 0 \le v \le \pi$

$\bar{r}_u = \langle \cos v,\ \sin v,\ 0 \rangle$

$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 1 \rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v,\ -\cos v,\ \underbrace{uc^2v + us^2v}_{\overset{\shortparallel}{u}} \rangle$$

$$|r_u \times r_v| = \sqrt{\sin^2 v + (-\cos v)^2 + u^2}$$
$$= \sqrt{u^2+1}$$

$$\int_{u=0}^{1}\int_{v=0}^{\pi} \sqrt{u^2+1}\, dv\, du \qquad = \pi \int_0^1 \sqrt{u^2+1}\, du$$

TRIG

$$\int \sqrt{u^2+a^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right)$$

$= (\text{us})$

$$\pi \cdot \left(\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2}\ln\left(u + \sqrt{u^2+1}\right) \right)\Bigg|_{u=0}^{1}$$

COOL DOWN

PRAC. TEST PROBLEMS

$$\oint_C \underbrace{\left(3y - \tan^{-1}(x^6)\right)}_{P} dx + \underbrace{\left(7x - \cos\sqrt{y-1}\right)}_{Q} dy$$

C: unit circle.

OF COURSE, GREEN

$Q_x = 7$

$P_y = 3$

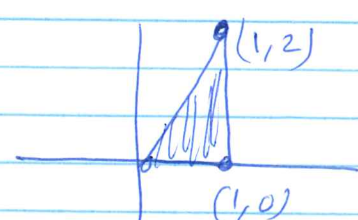
$$= \iint_{\text{unit circle}} (7-3)\, dA = \boxed{4 \cdot \pi}$$

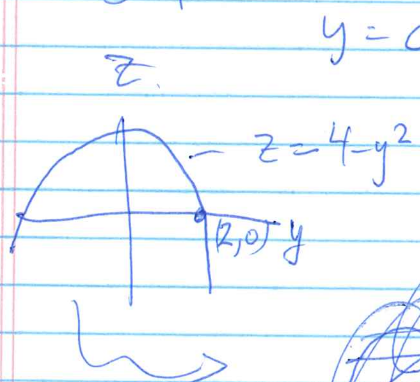
Rewrite $\displaystyle\int_0^1 \int_0^{2x} \int_0^{4-y^2} f(x,y,z)\, dz\, dy\, dx$

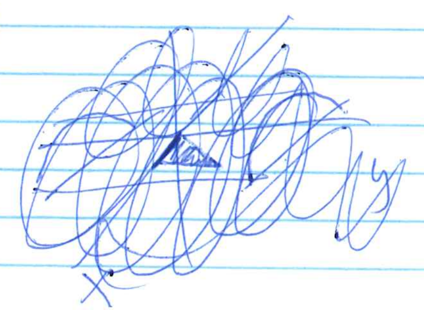
as $\iiint f \; dx\, dy\, dz$.

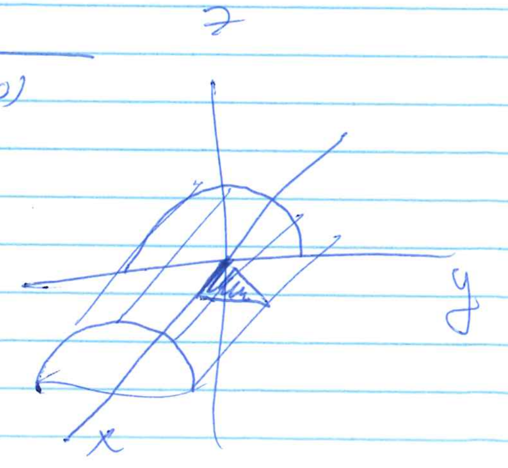
x-y plane $x = 0..1$

$y = 0..2x$









Throw Spear along innermost axis,

hits y-z plane in

$x = y/2$

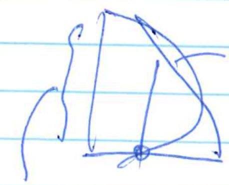
exits at $y = 2x$

hits where $x = 1$

Centry

So x goes from

$$\int_{z=0}^{4}\int_{y=0}^{\sqrt{4-z}}\int_{y/2}^{1} f \, dx$$



$z = 4 - y^2$

$z = 0..4$

$y^2 = 4 - z$

$y = \sqrt{4-z}$

Hard! Do Reps!