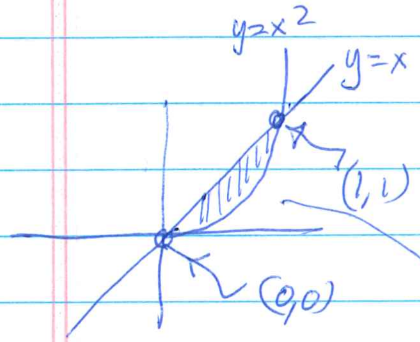
16.7, 16.8

(0)

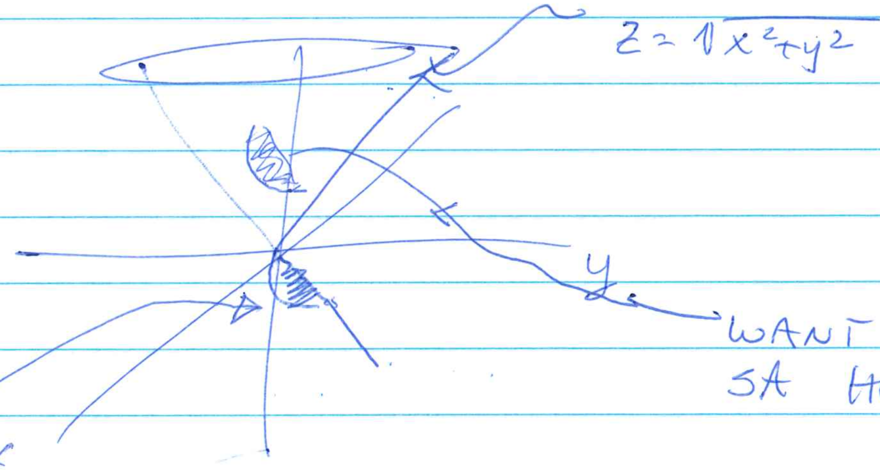
WARM UP Recall $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$$SA = \iint_D |r_u \times r_v| \, du\,dv \quad \star$$

16.6.42 Find Surface Area of part of cone $z = \sqrt{x^2+y^2}$ between $y = x$ and $y = x^2$.



PICTURE IN X–Y PLANE



Parameterization
$$x = x$$
$$y = y$$
$$z = (x^2+y^2)^{1/2}$$

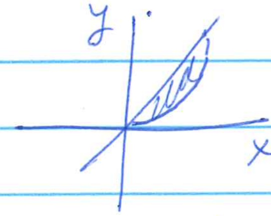
$$r_x = \langle 1, 0, \tfrac{1}{2}(x^2+y^2)^{-1/2}(2x) \rangle$$
$$r_y = \langle 0, 1, \tfrac{1}{2}(x^2+y^2)^{-1/2}(2y) \rangle$$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix}$$

$$= \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1 \right\rangle$$

Now $|r_u \times r_v| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1^2} = \sqrt{2}$!

So we get $\iint \sqrt{2}\, dx\, dy$



$$= \int_{x=0}^{1} \int_{y=x^2}^{x} \sqrt{2}\, dy\, dx$$

$$= \sqrt{2} \int_0^1 x - x^2\, dx = \sqrt{2} \left. \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1 \boxed{= \frac{\sqrt{2}}{6}}$$

How do we integrate a <u>function</u> on a surface?

On a curve, we did $\int_C f(x,y,z)\, ds$ — little piece of <u>arc</u>

$$= \int_C f(x,y,z) \cdot \underbrace{\sqrt{(x')^2 + (y')^2 + (z')^2}}_{|r'(t)|}\, dt$$

Plug parameterization in to f

i.e. $f(\underbrace{x(t), y(t), z(t)}_{\vec{r}(t)})$

$r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

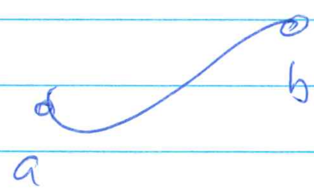
For surfaces, exactly same

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v)) \cdot |r_u \times r_v|\, du\, dv$$

§16.7: Oriented Surfaces

Just as it makes a difference which way we orient a curve when calculating work

$$W = \int_C \vec{F}(\vec{r}(t)) \cdot r'(t)\, dt \qquad \begin{cases} r(t) = (x(t), y(t), z(t)) \\ F = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle \end{cases}$$

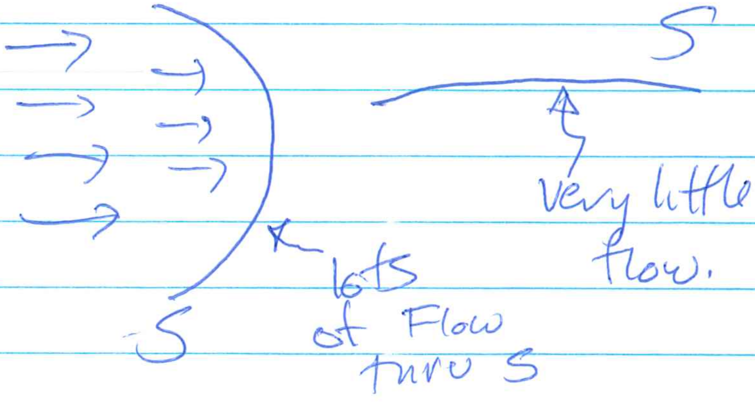


$$\int_{C\ a \text{ to } b} = -\int_{C\ b \text{ to } a}$$

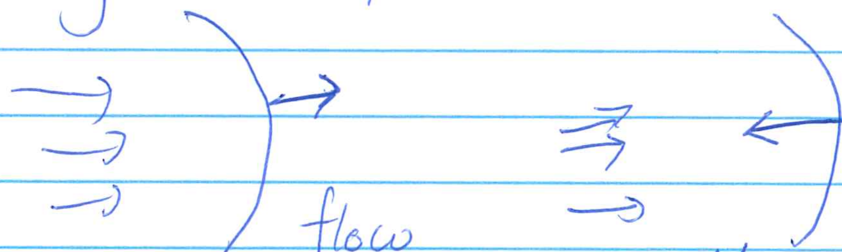
It will make a difference when we $\int$ a Vector field on a surface.

Q: What does this even mean?

A: Think of a surface as permeable then VF will have some "flow" thru surface



But <u>wait</u>: we need to say which way is "up"



"<u>orientation</u>" of a surface
<u>just</u> <u>like</u> choosing $\int_a^b$ or $\int_a^b$

When we did work, we had $\underbrace{F(r(t))}_{\text{vect field.}} \cdot \underbrace{r'(t)}_{\text{velocity}} dt$

It is <u>almost</u> exactly the same

size of path

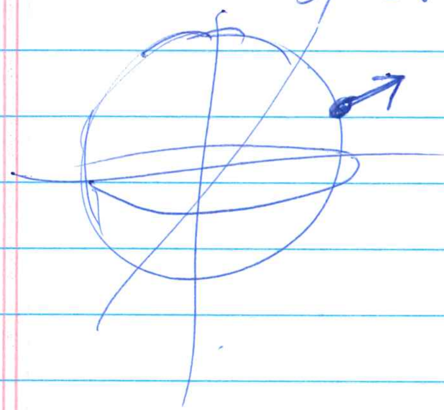
$$\iint_{\text{Surface}} \vec{F} = \iint F(\vec{r}(u,v)) \cdot \underbrace{\frac{r_u \times r_v}{|r_u \times r_v|}}_{\substack{\text{unit vector} \\ \text{giving} \\ \text{"up/down"}}} |r_u \times r_v|$$

Surface param. as $r(u,v)$ ($u,v \in D$)

$$\boxed{= \iint_D F(r(u,v)) \cdot \overbrace{\langle r_u \times r_v \rangle \, du\, dv}^{d\vec{S}}}$$

Example 4

Find flux of $\vec{F} = \langle z, y, x \rangle$ across unit sphere, w/ outward normal



unit sphere: ($\rho = 1$)

$x = \sin\phi \cos\theta$

$y = \sin\phi \sin\theta$

$z = \cos\phi$

$$r_\phi \times r_\theta = \begin{vmatrix} i & j & k \\ c\phi c\theta & c\phi s\theta & -s\phi \\ -s\phi s\theta & s\phi c\theta & 0 \end{vmatrix}$$

$d\vec{S} =$

$$\langle s^2\phi c\theta, s^2\phi s\theta, \underbrace{c\phi s\phi c^2\theta + c\phi s\phi s^2\theta}_{= c\phi s\phi} \rangle \quad (\text{UGH!})$$

$$\langle c\phi, s\phi s\theta, s\phi c\theta \rangle = F(r(\phi,\theta))$$

equal!

$$\boxed{s^2\phi c\phi c\theta + s^3\phi s^2\theta + s^2\phi c\phi c\theta} \quad ☆$$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} ☆ \, d\phi \, d\theta = 2\iint s^2\phi c\phi c\theta$$

$$+ \iint (1 - c^2\phi) s\phi s^2\theta$$

5

$$2\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} s^2\phi\, c\phi\, c\theta = 2\int \underbrace{\frac{s^3\phi}{3}\Big|_0^{\pi}}_{0} c\theta$$

This is done

$$+ \iint \underbrace{(1-c^2\phi)}\, s\phi\, \underline{\underline{s^2\theta}}$$

$$\int_{\theta=0}^{2\pi} \underbrace{\Big|_0^{\pi} -\cos\phi + \frac{\cos^3\phi}{3}}$$

$$\overbrace{1-\frac{1}{3}}^{\text{top}} - \overbrace{\left(-1+\frac{1}{3}\right)}^{\text{bot}} = \frac{4}{3}$$

$$\frac{4}{3}\int_{\theta=0}^{2\pi} \sin^2\theta\, d\theta = \frac{4}{3}\int_0^{2\pi} \frac{1-\cos 2\theta}{2}$$

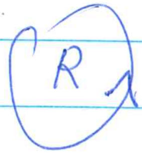
$$= \frac{4}{3}\cdot\frac{1}{2}\underbrace{\Big|_0^{2\pi} \theta - \frac{\sin 2\theta}{2}}_{2\pi}$$

$$= \boxed{\frac{4\pi}{3}}$$

LONG!

Greens Thm

$$\int_{R} P\,dx + Q\,dy = \iint_{R} Q_x - P_y \; dx\,dy$$



$\Downarrow$

$F = (P, Q)$

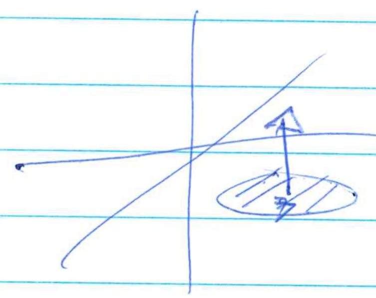
$r = (x(t), y(t))$

So above is $F \cdot dr$

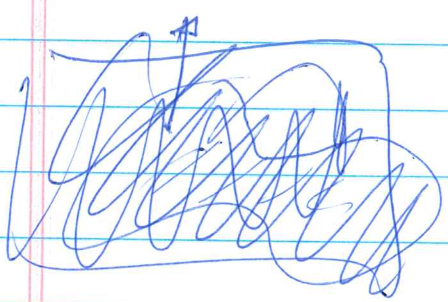
$F = \langle P, Q, 0 \rangle$

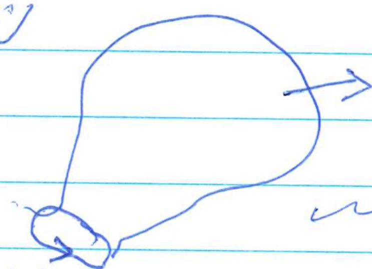
$Q_x - P_y = \text{curl} F \cdot k$

Normal to R in 3D.



STOKES is 3D GREEN





$\rightsquigarrow$ release air, ~~balloon~~ collapses

$\int$ around curve

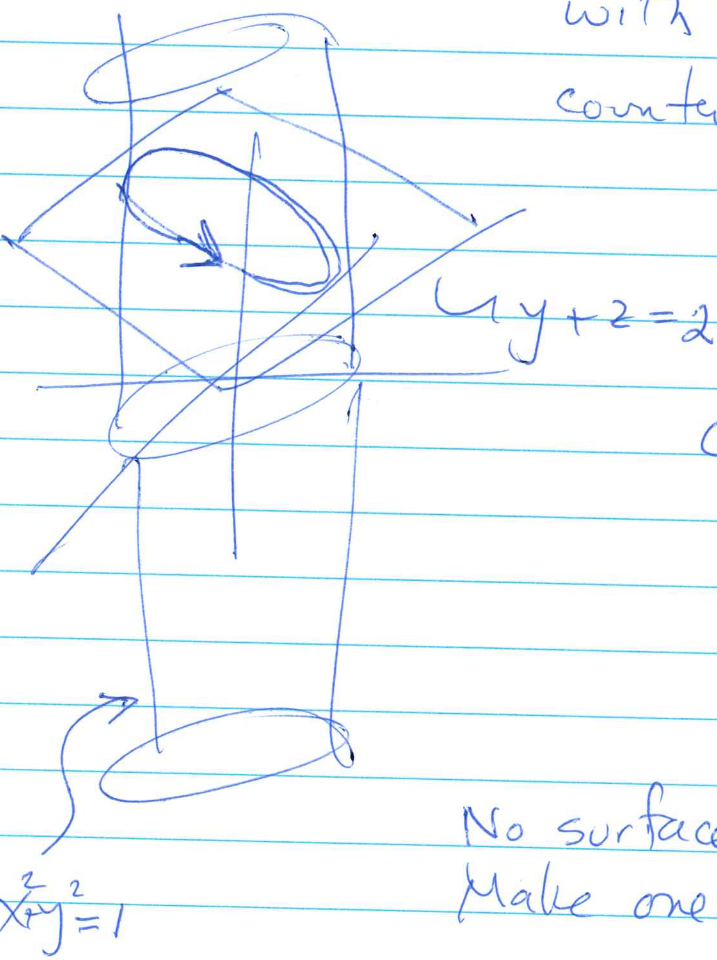
STOKES

$$\boxed{\int_C F \cdot d\vec{r} = \iint \text{curl} F \cdot d\vec{S}}$$

Example 1 (Book)

$$\int_C F \cdot dr, \quad F = \langle -y^2, x, z^2 \rangle$$

C = intersection of $y + z = 2$ with $x^2 + y^2 = 1$, oriented counterclockwise viewed from above.



Can do directly, but lets try **STOKES**

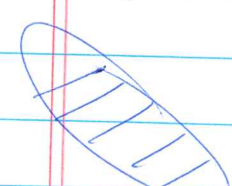
$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \langle 0, 0, 1 + 2y \rangle$$

(nice!)

No surface!! ☹

Make one by filling in



parameters: $x = x$, $y = y$, $z = 2 - y$

limits: unit disk in x-y plane

$$r(x,y) = \langle x, y, \underset{f(x,y)}{\overset{\shortparallel}{2 - y}} \rangle$$

$r_x = \langle 1, 0, 0 \rangle$

$r_y = \langle 0, 1, -1 \rangle$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \boxed{k}$$

So Stokes says $\quad r_x \times r_y = k$

$$\iint_{\text{unit disk}} \underbrace{\langle 0, 0, 1+2y \rangle}_{\text{curl } F} \circ \langle 0,0,1 \rangle \, dx\, dy$$

$$= \iint 1 + 2y \, dx\, dy$$

$$= \pi + 2 \iint y \, dy\, dx$$

$$= \pi + 2 \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\sin\theta \, r\, dr\, d\theta$$

$$= \pi + \frac{2}{3} \underbrace{\Big|_0^{2\pi} -\cos\theta}_{0}$$

Directly

$r(t): (\cos\theta, \sin\theta, 2-\sin\theta)$

$r't = (-s\theta, c\theta, -c\theta)$

$F(r(t)) =$
$\langle -s^2, c\theta, (2-\sin\theta)^2 \rangle$

$F(r(t)) \cdot r'(t) = -s^3 + c^2 - (2-\sin\theta)^2 c\theta$

Not too bad, finish and verify Stokes.