$\boxed{\begin{array}{l}\text{Final } 4/30 \\ 7\text{pm} - 930\text{pm}\end{array}}$

LAST LECTURE

0

16.9

$d\vec{S} = r_u \times r_v \, du\, dv$

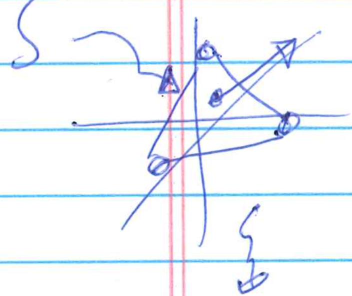
WARM UP

16.7/ $\int_S F \cdot d\vec{S}$ $\quad \vec{F} = \langle xze^y, -xze^y, z \rangle$

S = part of $x+y+z=1$ in first octant, oriented $\boxed{\text{downwards}}$!

① parametrize S: $x=x$, $y=y$, $z=1-x-y$

② compute $d\vec{S} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$

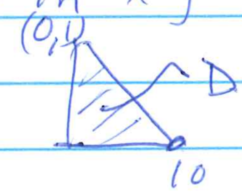


$\langle 1,1,1 \rangle \Leftarrow$

pointing wrong way!

$\Rightarrow$ use $\langle -1,-1,-1 \rangle$

Shadow in x-y plane is



plug in parametrization

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \langle x(1-x-y)e^y, -x(1-x-y)e^y, 1-x-y \rangle \cdot \langle -1,-1,-1 \rangle$$

CANCEL!

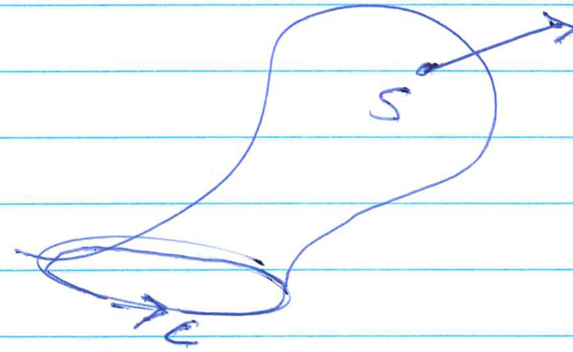
$$= \iint -(x(1-x-y)e^y) + x(1-x-y)e^y \, (+x+y-1) \, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (x+y-1) \, dy\, dx$$

$$= \int_{x=0}^{1} \left[\left(xy + \frac{y^2}{2} - y \right) \right]_{y=0}^{1-x} dx$$

$$\int_{x=0}^{1} x(1-x) + \frac{(1-x)^2}{2} - (1-x) \, dx = \text{Calc II.}$$

Remember Stokes:



ORIENTED LIKE GREEN'S THM

RHR

$$\int_C F \cdot dr = \iint_S (\text{curl } F) \cdot d\vec{S}$$ "open surface"

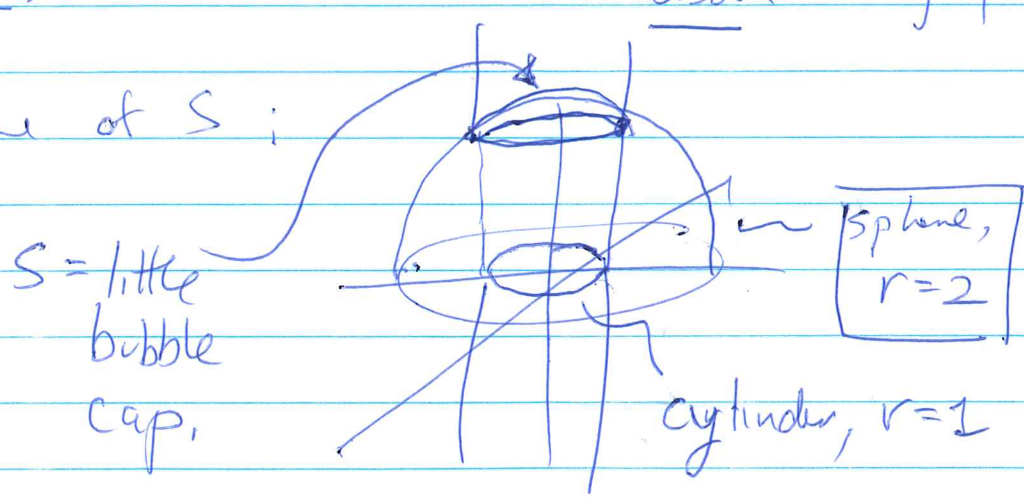
~~LAP 2~~ Compute $\iint_S \cancel{F \cdot d\vec{S}} \; \text{curl}(F) \cdot d\vec{S}$

where $F = \langle xz, yz, xy \rangle$ and

$S = \{x^2 + y^2 + z^2 = 4\}$ inside $x^2 + y^2 = 1$ above x-y plane.

ORIENTED UP

(1) Picture of S:



2

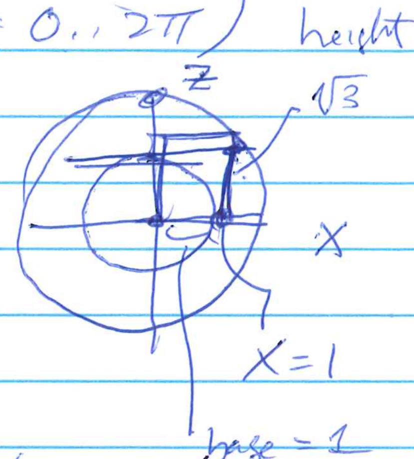
parameterize S: $x = x,\ y = y,\ z = \sqrt{4-x^2-y^2}$

$\boxed{\text{OR}}$ $(\rho = 2)$

$x = 2\sin\phi\cos\theta$
$y = 2\sin\phi\sin\theta$
$z = \cos\phi$

what is ϕ limit? $(\theta = 0..2\pi)$

look at from side



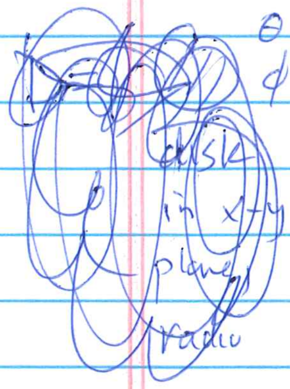
$x^2+y^2+z^2 = 4$
$x^2+y^2 = 1$
$\Rightarrow z^2 = 3 \Rightarrow z = \sqrt{3}$

$\Rightarrow$ this angle is 60, so $\phi = 30 = \underline{\underline{\pi/6}}$

plug in

$$\iint \langle 2s\phi c\theta c\phi,\ 2s\phi s\theta c\phi,\ 4s^2\phi c\theta s\theta \rangle \cdot \langle \qquad \rangle$$

$\theta = 0..2\pi$
$\phi = 0..\pi/6$



$$\begin{array}{c} \\ r_\phi \\ r_\theta \end{array} \begin{vmatrix} i & j & k \\ +2c\phi c\theta, & 2c\phi s\theta, & -s\phi \\ -2s\phi s\theta, & 2s\phi c\theta, & 0 \end{vmatrix}$$

$$= \langle 2s^2\phi c\theta,\ 2s^2\phi s\theta,\ 4c\phi s\phi \rangle$$

Dot product (✱) is

$$\underbrace{4s^3\phi c^2\theta c\phi + 4s^3\phi s^2\theta c\phi}_{4s^3\phi} + 16s^3\phi c\phi\ (c\theta s\theta)$$

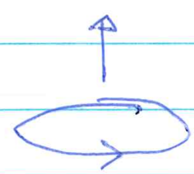
$$\iint 4\sin^3\phi \,(\text{easy}) + \iint \underbrace{16\sin^3\phi\cos\phi}_{4\sin^4\phi} \underbrace{\cos\theta\sin\theta}_{s^2\theta}$$

$$= 4(1-c^2)s\phi$$

$$= -4c\phi + 4\frac{c^3\phi}{3}$$

HARD!

BUT, WHEN WE LOOKED AT IT, WHAT IF WE TRIED THE $\int_C$ FIRST?



C is a unit circle, at height $z = \sqrt{3}$

$\Rightarrow C = (\cos t, \sin t, \sqrt{3})$ "$\vec{r}(t)$" $t = 0..2\pi$

$$\int_{t=0}^{2\pi} \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos t\sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

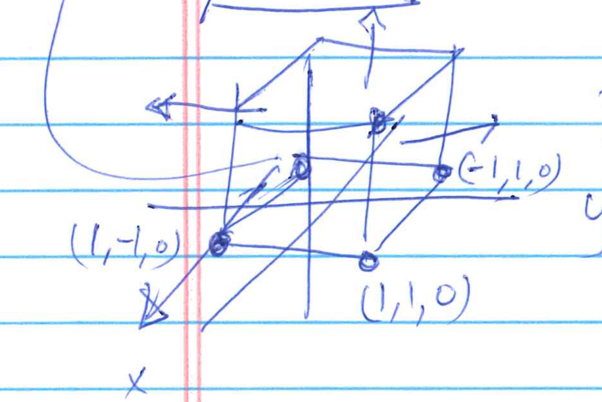
CANCEL!

$$= \int_0^{2\pi} -\sqrt{3}\cos t\sin t + \sqrt{3}\cos t\sin t + 0 \, dt$$

$$= 0$$

SWEET!

LAP 3



Box, open bottom, vertices at (1,1)(-1,1)(1,-1)(-1,-1)

oriented outward.

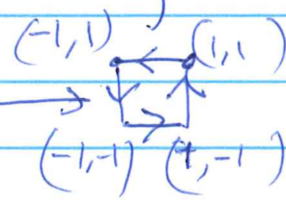
$$\text{Find } \iint_S \text{curl}(F)\cdot d\vec{S} = \star$$

$$F = \langle xyz,\ xy,\ x^2yz \rangle.$$

$\boxed{\text{Double stokes trick}}$

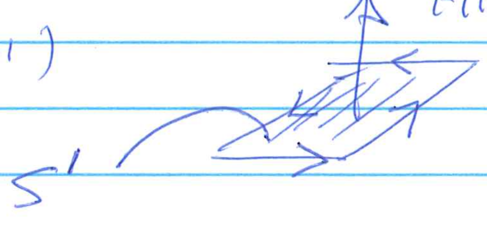
$$\star = \int F\cdot dr = \iint$$

CURVE ⟶



IN

x-y plane



Filled in square.

Notice $d\vec{S} = \langle 0,0,1\rangle$

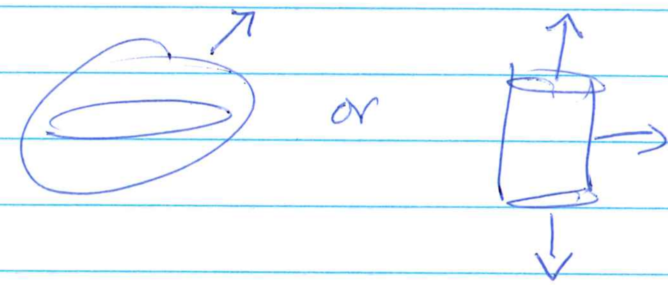
So when we do $\text{curl } F\cdot d\vec{S}$, only need z-part!

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xyz & xy & x^2yz \end{vmatrix} \xrightarrow{k} y - xz \quad \text{but on } S',\ z=0!$$

$$= \int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy = \int \frac{y^2}{2}\Big|_{-1}^{1} dx = 0!$$

16.9 Divergence (Easy Wrap Up)

S a closed surface, outward pointing normal



Then $$\iint_S F \cdot d\vec{S} = \iiint_{\substack{\text{inside of } S \\ = V}} \operatorname{div}(F) \cdot dV.$$

Example: $\vec{F} = \langle x, y, z \rangle$, S = unit sphere

Check Stokes

$\operatorname{div} F = 1 + 1 + 1 = 3$

$$\iiint_{\text{unit sphere}} 3\,dV = 3 \cdot \text{Vol unit sphere} = 3 \cdot \frac{4}{3}\pi = \boxed{4\pi}$$

DO FIRST

Param sphere:

$\vec{r} = \langle \sin\phi\cos\theta, s\phi s\theta, c\phi \rangle$

$$\begin{vmatrix} i & j & k \\ c\phi c\theta, & c\phi s\theta, & -s\phi \\ -s\phi s\theta, & s\phi c\theta & 0 \end{vmatrix} = \langle s^2\phi c\theta, s^2\phi s\theta, c\phi s\phi \rangle$$

$\cdot \langle s\phi c\theta, s\phi s\theta, c\phi \rangle$

$= s^3\phi\,(c^2\theta + s^2\theta) + c^2\phi s\phi$

$$\text{So} \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \left(s^3\phi + c^2\phi\, s\phi\right) d\phi\, d\theta$$

$$\int \left(1-c^2\phi\right) s\phi + c^2\phi\, s\phi$$

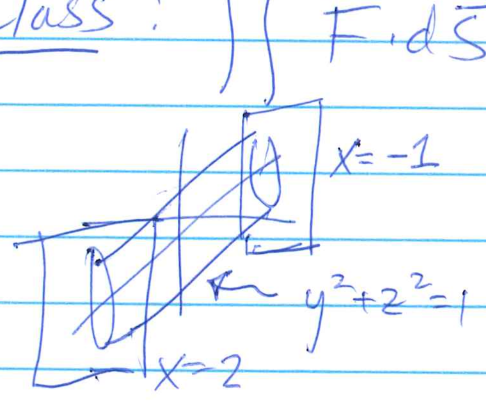
$$\downarrow \qquad\qquad \downarrow$$

$$-c\phi + \underbrace{\frac{c^3\phi}{3} \qquad -\frac{c^3\phi}{3}}_{\text{cancel.}}$$

$$\int_0^{2\pi} \left(-\cos\phi \Big|_{\phi=0}^{\pi} \right) d\theta$$

$$= -\Big[(-1) - 1\Big] = 2\int_0^{2\pi} d\theta = \boxed{4\pi}$$

<u>Class</u>: $\iint \vec{F} \cdot d\vec{S}$

$S =$ 

Notice: S has 3 parts: it is a can, with top, bottom, sides.

$$F = \langle 3xy^2, xe^z, z^3 \rangle$$

Lets try Divergence Thm.

$$\operatorname{div} F = 3y^2 + 0 + 3z^2.$$

$$\iiint 3(y^2+z^2)\, dV$$

well our region is $x=-1..2$

and $\underbrace{y^2+z^2}\leq 1$.

$y = r\cos\theta$

$z = r\sin\theta$

$\theta = 0..2\pi \quad r=0..1$

So we use

$$3\int_{x=-1}^{2}\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2 (r\,dr\,d\theta)\,dx \qquad \boxed{y^2+z^2=r^2}$$

$$= \frac{r^4}{4}\Big|_0^1 \cdot \Big|_0^{2\pi}\theta \;\Big|_{-1}^{2} x$$

$$= 3\cdot\frac{1}{4}\cdot 2\pi\cdot 3 = \boxed{\frac{9\pi}{2}}$$

easy peasy!