Lecture 1

MATH OF PAGE RANK

Class 1
- Linear equations ①
- Gaussian elim ②
- Matrix multiplication ③
- Markov process ④

① How do we solve linear eqns?

$$x - 2y = 2$$
$$2x + 3y = 6$$

Soln: (Gaussian elim) Try to reduce to a system in a smaller # of variables.

$$(-2)\ (x - 2y = 2)$$
$$+(1)\ (2x + 3y = 6)$$
$$0 + 7y = 2$$

DRAW ⟹ PIC ..

$\Rightarrow y = 2/7$, backsolve

$x - 2(2/7) = 2$ so

$x = 18/7.$

Check: $18/7 - 2(2/7) = 14/7 = 2$ ✓

Algorithm:
$$a_{11}x_1 + \cdots\ a_{1n}x_n = b_1$$
$$a_{21}x_1 + \cdots\ a_{2n}x_n = b_2$$
$$\vdots$$

Lecture 1

(1)

- Swap order of eqns so $a_{11} \neq 0$.
- use a_{11} to kill entries below it.
- left with a system of fewer eqns
- in fewer unknowns.

Class Solve

$$\begin{aligned} x + y + z &= 3 \\ 2x + y \quad &= 7 \\ 3x + \quad 2z &= 5 \end{aligned}$$

$$\begin{aligned} x + y + z &= 3 \\ 0 \quad -y - 2z &= 1 \\ 0 \quad -3y - z &= -4 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} -y - 2z &= +1 \\ 0 \quad 5z &= -7 \end{aligned}$$

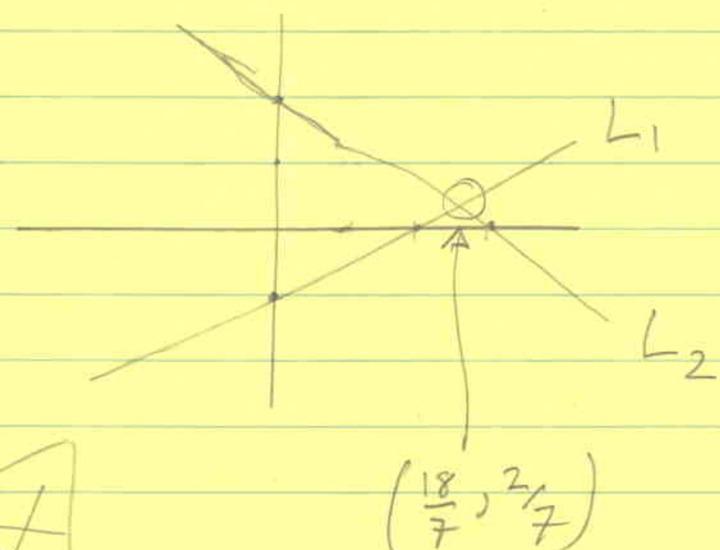
$z = -7/5$ backsolve

Geometry

L_1 $x - 2y = 2$

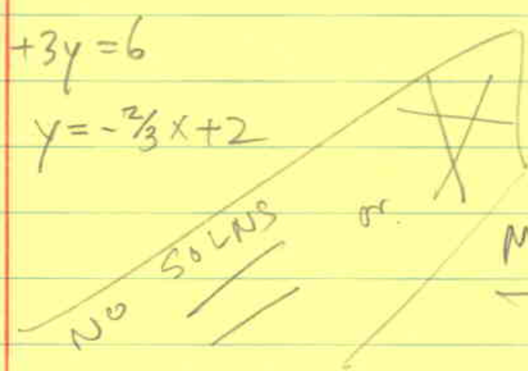
$2y = x - 2$

$y = \frac{1}{2}x - 1$



L_2 $2x + 3y = 6$

$y = -\frac{2}{3}x + 2$



Mention: 3 poss:
- No solns
- 1 soln
- ∞ solns.

Shorthand

A matrix is an array of numbers

$$\begin{bmatrix} 2 & 7 \\ 3 & 3 \\ 1 & 5 \end{bmatrix}.$$

Rule for multiplying

If A is $m \times n$ and B is $n \times k$

rows cols rows cols.

then we can multiply

AB is an $n \times k$ matrix

with $(AB)_{ij}$ ← entry in row i col j

given by $\text{Row}_i(A) \cdot \text{Col}_j(B)$

$$(\alpha_1 \ldots \alpha_n) \cdot (\beta_1 \ldots \beta_n) = \sum \alpha_i \beta_i$$

example 1

$$\underset{3\times 2}{\begin{bmatrix} 2 & 7 \\ 3 & 3 \\ 1 & 5 \end{bmatrix}} \cdot \underset{2\times 4}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}} = \overset{3\times 4}{\begin{bmatrix} 37, & 46, & 55, & 64 \\ . & . & . & . \\ . & . & . & . \end{bmatrix}}$$

ex 3 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

ID MATRIX

example 2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4 & & \\ & 10 & \\ & & 18 \end{bmatrix}$$

DIAGONAL MATS ARE NICE !!!

Notice: We can now write our linear equations as e.g

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

WHO CARES? A VIGNETTE:

In Smallville 30% nonsmokers $\longrightarrow$ smokers
20% smokers $\longrightarrow$ quit.

If at $t=0$ we have 8000 smokers
2000 nonsmokers

What are the # at 100 yrs? 1000 years?

Lets take a baby step, and see what happens at $t=1$.

$$NS(1) = .7\,NS(0) + .2\,S(0)$$
$$S(1) = .3\,NS(0) + .8\,S(0)$$

$$\begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix}\begin{pmatrix} NS(0) \\ S(0) \end{pmatrix} = \begin{pmatrix} NS(1) \\ S(1) \end{pmatrix}.$$

How about at $t=2$.?

$$\begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix}\begin{pmatrix} N(1) \\ S(1) \end{pmatrix} = \begin{pmatrix} N(2) \\ S(2) \end{pmatrix}$$

So, if we call $A = \begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix}$

We have $\begin{pmatrix} N(k) \\ S(k) \end{pmatrix} = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k\text{-times}} \begin{pmatrix} N(0) \\ S(0) \end{pmatrix}$

<u>Problem</u> multiplying A^n, for n large, is <u>costly</u>

<u>Deus Ex Machina</u>

$\begin{vmatrix} 7-\lambda & 2 \\ 3 & 8-\lambda \end{vmatrix}$

$\lambda^2 - 15\lambda + 50$

$(\lambda-5)(\lambda-10)$

evects

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\lambda = 5$

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\lambda = 10$

$$\underbrace{\frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}}_{B^{-1}} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

$$B \cdot B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{(BAB^{-1})(BAB^{-1}) \ldots (BAB^{-1})}_{n \text{ times}} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}^n$$

$$B A^n B^{-1} = D^n$$

$$A^n = B^{-1} D^n B.$$