Lecture 3

MATH OF PAGE RANK

Last Time: Defined basis and Δ basis

TODAY (1) Linear Transforms
(2) Rep. via matrices
(3) Eigenvalues & Eigenvectors.

Def: A map $V \xrightarrow{L} V$ is a linear transform if (1) $L(a\vec{v}) = aL\vec{V}$
(2) $L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2)$

"sums split up, scalars pull out".

Thm: Any Linear Transform can be represented by matrix multiplication. The entries of matrix depend on choice of basis.

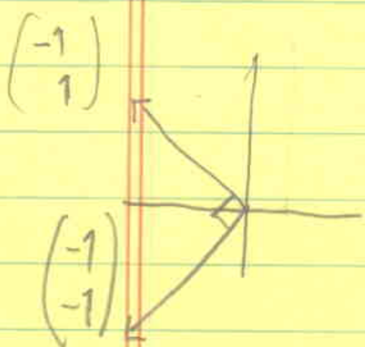
ALGORITHM given a basis $B = \{\vec{v}_1 .. \vec{v}_n\}$ and linear transform L, the matrix representing L wrt basis B is

$$\left[L(\vec{v}_1)_B \,\middle|\, \cdots \,\middle|\, L(\vec{v}_n)_B \right]$$

example: L rotates $90°$ counterclockwise
suppose B is $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

$\begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}-1\\1\end{pmatrix}$ ← in Basis $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$

this is $\begin{pmatrix}0\\1\end{pmatrix}_B$!



$\begin{pmatrix}-1\\1\end{pmatrix} \to \begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}_B.$

check: $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix}_{B_1} = \begin{pmatrix}0\\1\end{pmatrix}_{B_1}$ ✓

Class

Write out the matrix for L w.r.t basis $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$. B'

Soln $\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$\begin{pmatrix}1\\1\end{pmatrix} \to \begin{pmatrix}1\\-1\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$

So $L_{B'} = \begin{bmatrix}-1 & 2\\1 & -1\end{bmatrix}$

Check: $\begin{bmatrix}-1 & 2\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_{B_1} = \begin{bmatrix}-1\\1\end{bmatrix}_{B_1} = -\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

FLASHBACK Day 1, we found matrices

So $B^{-1}AB = D$ ← diagonal.

What really happened:

$$A = [L]_{B_1} \quad , \quad D = [L]_{B_2} .$$

B is our <u>translator</u> matrix

$$\Delta_{B_1 \to B_2} [L]_{B_1} \Delta_{B_2 \to B_1} = [L]_{B_2}$$

— give "translator" analogy —

<u>Def</u>: $\vec{v}_1$ is an eigen vector for a matrix A if $A\vec{v}_1 = \underset{\substack{\uparrow \\ \text{const.}}}{c_1}\vec{v}_1$

<u>idea</u> we're gonna find a basis $\overset{B_E}{\{\vec{v}_1 .. \vec{v}_k\}}$ of eigen vectors, because then $[A]_{B_E} = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_k \end{bmatrix}$

<u>ALGORITHM</u> $A\vec{v}_1 = c\vec{v}_1 \iff (A - cI)\vec{v}_1 = 0$

1. eigenvals are roots of $\det [A - cI]$
2. eigen vectors are solns to the system

<u>Example</u> $B = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

$$L_B = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix} = A$$

<u>Step 1</u> $\det \underbrace{\begin{bmatrix} 7-c & 2 \\ 3 & 8-c \end{bmatrix}}_{\parallel} = (7-c)(8-c) - 2\cdot 3$

(Here we only do det of a 2×2, the general def is recursive, det A an $n\times n$ matrix is $\underbrace{\sum (-1)^i a_{ji} A_{ji}}_{\parallel}$, where A_{ji} = det of cross out row j, col i)

Anyhow, $(7-c)(8-c) - 2\cdot 3 = c^2 - 15c + 50$
$= (c-5)(c-10)$

* eigenvalues are 5, 10 *

$c = 5$: solve $\begin{bmatrix} 7-5 & 2 \\ 3 & 8-5 \end{bmatrix} \vec{v} = \begin{pmatrix}0\\0\end{pmatrix}$

$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \vec{v} = \begin{pmatrix}0\\0\end{pmatrix}$, any multiple of $\begin{pmatrix}1\\-1\end{pmatrix}$

$c = 10$: solve $\begin{bmatrix} 7-10 & 2 \\ 3 & 8-10 \end{bmatrix} \vec{v} = \begin{pmatrix}0\\0\end{pmatrix}$ so $\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \vec{v} = \begin{pmatrix}0\\0\end{pmatrix}$, soln $\begin{pmatrix}2\\3\end{pmatrix}$

Our basis of eigenvectors is $\underbrace{\left\{ \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix} \right\}}_{\parallel}$

B_2

HOMEWORK $B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, $B_2 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$ 4

1 WHAT IS $\Delta_{B_1 \to B_2}$??

solve $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

2 WHAT IS $\Delta_{B_2 \to B_1}$??

solve $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\boxed{\text{easy}}$ $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

3 COMPUTE $\Delta_{B_1 \to B_2} A \, \Delta_{B_2 \to B_1}$.

To compute $\Delta_{B_1 \to B_2}$, solve both systems simultaneously:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array} \qquad \begin{array}{cccc} 1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}$$

$$\Delta_{B_1 \to B_2} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

check:

$$\underbrace{\frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}}_{\Delta_{B_1 \to B_2}} \underbrace{\begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix}}_{A \,=\, [L]_{B_1}} \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}}_{\Delta_{B_2 \to B_1}}$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 20 \\ -5 & 30 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 0 \\ 0 & 50 \end{bmatrix}$$

$$\cdot \frac{1}{5} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

MYSTERY MATRIX
D FROM
DAY 1 !!!