

CALC III STUDENTS

PRACTICE FINAL PROBLEMS

TEST : 7PM → 930 PM 4/30

OPEN Book + NOTES, NO ELECTRONICS.

GUARANTEED

1 PROBLEM

ON →

EACH OF THESE
TOPICS

- MAX / MIN CRITICAL PTS
- STOKES
- DIVERGENCE
- SOME TYPE OF LINE INTEGRAL

TOTAL WILL BE 7 - 8 PROBLEMS

FINAL IS 25% OF GRADE.

PROBLEMS HERE
ARE TO GIVE
IDEA OF DIFFICULTY
FINAL WILL NOT
MIRROR THESE

HINT: COPY PROBLEMS AND WORK THEM
IN A TEST ENVIRONMENT. DON'T
LOOK AT SOLNS UNTIL YOU'VE
WORKED A COUPLE HOURS.

STUDY HARD!
H.

Consider the function

$$f(x, y) = 4x^2 + y^4 - 7y^2 - 4xy + 4y - 8x + 10.$$

Find all critical points of the function f and classify them.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

$$0 = \frac{\partial f}{\partial x} = 8x - 4y - 8 = 0 \quad \leftarrow \text{solve for } x, \text{ then substitute}$$

$$0 = \frac{\partial f}{\partial y} = 4y^3 - 14y - 4x + 4 = 0 \quad \leftarrow \quad 4x = 2y + 4$$

$$0 = 4y^3 - 14y - (2y + 4) + 4 = 4y^3 - 16y = 4y(y^2 - 4) \\ = 4y(y-2)(y+2)$$

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$$= 8(12y^2 - 14) - 16$$

$$D(0, -2) > 0, \quad f_{xx} > 0 \Rightarrow \text{local min}$$

$$D(1, 0) < 0 \Rightarrow \text{saddle}$$

$$D(2, 2) > 0, \quad f_{xx} > 0 \Rightarrow \text{local min}$$

$$\begin{aligned} &\underbrace{= 0 \Leftrightarrow \text{some term is } 0}_{\text{so}} \\ &\text{so } y = 0, y = 2, y = -2 \\ &\downarrow \quad \downarrow \quad \downarrow \\ &x = 1 \quad x = 2 \quad x = 0 \end{aligned}$$

critical points are
 $(0, -2), (1, 0), (2, 2)$

Answer:

$(0, -2)$ a local min

$(1, 0)$ a saddle

$(2, 2)$ a local min

Find all critical points of the function

$$f(x, y) = xy + \frac{2}{x} + \frac{4}{y}$$

and classify each as local maximum, local minimum, or saddle point.

$$f_x = y - \frac{2}{x^2} = 0 \text{ or } y = \frac{2}{x^2}$$

$$f_y = x - \frac{4}{y^2} = 0 \text{ or } x = \frac{4}{y^2} = \frac{4}{(\frac{2}{x^2})^2} = x^4 \Rightarrow 0 = x^4 - x = x(x^3 - 1)$$

$x=0 \text{ or } x=1$

but $x=0$ is impossible
(not in domain)

only one critical point $x=1, y=2$

$$f_{xx} = \frac{4}{x^3} \quad D = f_{xx} f_{yy} - (f_{xy})^2 = \frac{32}{x^3 y^3} - 1$$

$$f_{xy} = 1 \quad D(1, 2) = \frac{32}{1 \cdot 8} - 1 = 3 > 0$$

$$f_{yy} = \frac{8}{y^3} \quad f_{xx}(1, 2) = 4 > 0$$

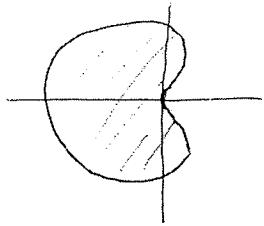
critical point at
(1, 2) is a MIN

Evaluate the line integral

$$\oint_C \underline{M} dx + \underline{N} dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

where the curve C is the cardioid given in polar coordinates by
 $r = 1 - \cos \theta, 0 \leq \theta \leq 2\pi$.

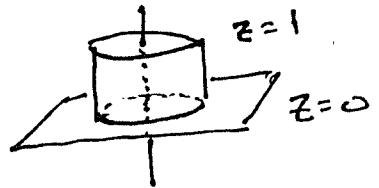
GREEN'S
THEOREM



$$\iint_S (2x - 2x) dA$$

$$\iint_S 0 dA$$





Consider the solid \mathcal{D} enclosed by $x^2 + y^2 = 1$, $z = 0$ and $z = 1$. Let

$$\mathbf{F}(x, y, z) = (xy^2 + e^{-z^2})\mathbf{i} + (x^2y + \arctan(xz))\mathbf{j} + (x^2y^2 + x^2z^2 + y^2z^2)\mathbf{k}$$

be a vector field defined on \mathcal{D} and S be the closed boundary surface of \mathcal{D} . Use Gauss's Divergence Theorem to calculate $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$.

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma &= \iiint_D (\operatorname{div} \mathbf{F}) dV \\
 &= \iiint_D \left(\underbrace{y^2 + x^2}_{r^2} + \underbrace{2zx^2 + 2zy^2}_{2zr^2} \right) dV \iff (2z+1)r^2 \\
 &= \int_0^{2\pi} \int_0^1 \int_0^1 (2z+1)r^2 \cdot r dz dr d\theta \\
 &= \underbrace{\left(\int_0^{2\pi} d\theta \right)}_{\theta \Big|_0^{2\pi} = 2\pi} \underbrace{\left(\int_0^1 r^3 dr \right)}_{\frac{1}{4}r^4 \Big|_0^1 = \frac{1}{4}} \underbrace{\left(\int_0^1 (2z+1) dz \right)}_{(z^2+z) \Big|_0^1 = 2} \\
 &= \boxed{\pi}
 \end{aligned}$$

Answer:

Find $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$ where $\mathbf{F} = \langle \sqrt{z}, -2x, \sqrt{y} \rangle$ and C is the parameterized curve from $(0, 0, 0)$ to $(2, 4, 16)$ given by (t, t^2, t^4) where $0 \leq t \leq 2$.

$$\begin{aligned}
 & \int_C \sqrt{z} dx - 2x dy + \sqrt{y} dz \\
 &= \int_0^2 (t^2 \cdot 1 - 2 \cdot t \cdot 2t + t \cdot 4t^3) dt \\
 &= \int_0^2 (4t^4 - 3t^2) dt \\
 &= \left(\frac{4}{5}t^5 - t^3 \right) \Big|_0^2 \\
 &= \left(\frac{128}{5} - 8 \right) - 0 \\
 &= \frac{128}{5} - \frac{40}{5} \\
 &= \boxed{\frac{88}{5}}
 \end{aligned}$$

score

Consider the function

$$f(x, y, z) = x^2 + 3xy - z^2 + 2y + z + 4.$$

- (a) Find the direction where the function f increases most rapidly at the point $(1, 1, 0)$.
Also compute the rate of increase along that direction.

↑ use gradient

$$\nabla f = \langle 2x+3y, 3x+2, -2z+1 \rangle$$

$$\nabla f(1, 1, 0) = \langle 5, 5, 1 \rangle$$

$$\|\nabla f(1, 1, 0)\| = \sqrt{25+25+1} = \sqrt{51} \leftarrow \text{rate of change}$$

Answer:

$$\text{Direction: } \frac{1}{\sqrt{51}} \langle 5, 5, 1 \rangle \quad \text{Rate of change: } \sqrt{51}$$

- (b) Find the directional derivative of f at the point $(1, 1, 0)$ in the direction of the vector $2i - j - 2k$.

$$\underbrace{\text{need to make into unit vector: }}_{\|\langle 2, -1, -2 \rangle\| = \sqrt{4+1+4}=3} \langle 2, -1, -2 \rangle$$

$$\begin{aligned}\nabla f(1, 1, 0) \cdot \frac{1}{3} \langle 2, -1, -2 \rangle &= \langle 5, 5, 1 \rangle \cdot \frac{1}{3} \langle 2, -1, -2 \rangle \\ &= \frac{1}{3} (10 - 5 - 2) \\ &= 1\end{aligned}$$

Answer:

1

- (a) Find an equation for the plane that passes through the point $(1, 2, 3)$ and perpendicular to the line $x = 1 + 3t$, $y = 3 + 2t$, $z = 1 - t$.

normal = direction
vector of line

$$= \langle 3, 2, -1 \rangle$$

$$0 = 3(x-1) + 2(y-2) \cancel{- (z-3)}$$

$$\text{or } 3x + 2y - z = 4$$

Answer:

$$3x + 2y - z = 4$$

- (b) Find the distance from the point $(3, 2, 1)$ to the plane from part (a).

Distance from (x_0, y_0, z_0) to plane $Ax + By + Cz = D$

$$\text{is } \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|3 \cdot 3 + 2 \cdot 2 - 1 \cdot 1 - 4|}{\sqrt{3^2 + 2^2 + (-1)^2}} = \frac{8}{\sqrt{14}}$$

Answer:

$$\frac{8}{\sqrt{14}}$$

(a) Is the vector field $\langle 2xe^{x^2} + z, 1 + e^y, x + 2z \rangle$ conservative? Justify your answer.

$f(x, y, z) = e^{x^2} + xz + y + e^y + z^2$ is a potential function,
so the vector field is conservative.

Alternate argument

$$\nabla \times \langle 2xe^{x^2} + z, 1 + e^y, x + 2z \rangle = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xe^{x^2} + z & 1 + e^y & x + 2z \end{vmatrix} = \begin{matrix} \partial i + j + \partial k \\ -\partial i - j - \partial k \end{matrix} = 0$$

↑
so vector field
is conservative

(b) Let $C = ((t-1)e^t, t + \cos(\frac{\pi}{2}t) \ln(1+t^2), (t+1)^{(t+1)} - 1)$ be a parametric curve with $0 \leq t \leq 1$. Find the following

$$\int_C \langle 2xe^{x^2} + z, 1 + e^y, x + 2z \rangle \cdot T ds.$$

We use potential function from (a)

$$C(0) = (-1, 0, 0) \leftarrow \text{start}$$

$$C(1) = (0, 1, 3) \leftarrow \text{end}$$

$$\int_C \langle 2xe^{x^2} + z, 1 + e^y, x + 2z \rangle \cdot T ds = (e^{x^2} + xz + y + e^y + z^2) \Big|_{(-1, 0, 0)}^{(0, 1, 3)}$$

$$= (1 + 0 + 1 + e + 9) - (e + 0 + 0 + 1 + 0)$$

$$= \boxed{10}$$

score

Let S be the upper hemisphere of $x^2 + y^2 + (z - 3)^2 = 9$ (the part with $z \geq 3$) and with upward pointing normal \mathbf{n} . Calculate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$ where $\mathbf{F} = -yz\mathbf{i} + xz\mathbf{j} + (z^3 - x^2y)\mathbf{k}$.

As a line integral (Stokes')

$$C = (3\cos t, 3\sin t, 3) \text{ for } 0 \leq t \leq 2\pi$$

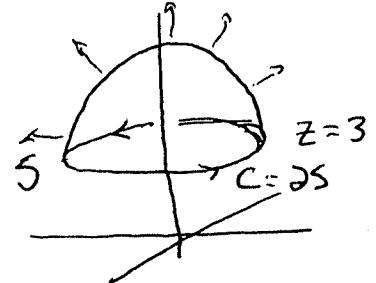
$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

$$= \int_C -yz dx + xz dy + (z^3 - x^2y) dz$$

$$= \int_0^{2\pi} [-(3\sin t) \cdot 3 \cdot (-3\sin t) + (3\cos t)(3)(3\cos t) + (3^3 - (3\cos t)^2(3\sin t)) \cdot 0] dt$$

$$= \int_0^{2\pi} (27\sin^2 t + 27\cos^2 t) dt$$

$$= \int_0^{2\pi} 27 dt = 27t \Big|_0^{2\pi} = \boxed{54\pi}$$



As a different surface (Stokes' + B.Y.O.S)

G^2 disk of radius 3 at $z = 3$
upward normal.

$$\partial S = C = \partial G$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_G (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$$

$$= \iint_G \langle -x^2 - y, -y + 2xy, 2z \rangle \cdot \langle 0, 0, 1 \rangle d\sigma$$

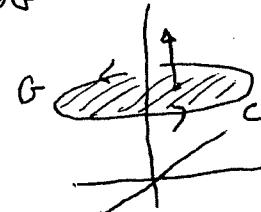
G

$$= \iint_G 2z \cdot d\sigma \quad \text{since } z = 3$$

$$= \iint_G 6 d\sigma \quad \text{surface area}$$

$$= 6(\text{area of } G) \quad \text{area of circle of radius 3}$$

$$= 6(9\pi) = \boxed{54\pi}$$



$$\mathbf{n} = \langle 0, 0, 1 \rangle$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz & xz & z^3 - x^2y \end{vmatrix}$$

$$= -x^2 i - y j + z k \\ -x i + 2xy j + z k$$

score

$$\therefore \langle -x^2 - y, -y + 2xy, z \rangle$$