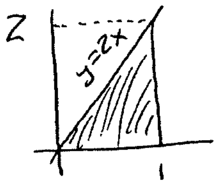
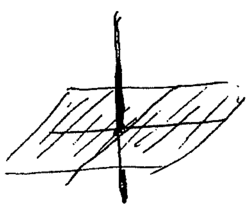
6. Rewrite the integral $\int_0^1 \int_0^{2x} \int_0^{4-y^2} f(x,y,z)\, dz\, dy\, dx$ as an iterated integral with order of integration dx dy dz.

$$\left.\begin{aligned} &0 \le x \le 1 \\ &0 \le y \le 2x \end{aligned}\right\} \rightarrow$$

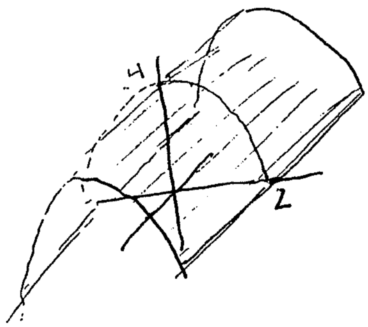


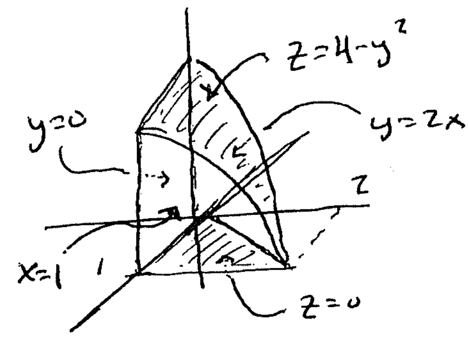
$$0 \le z \le 4-y^2$$

$z=0$
xy-plane

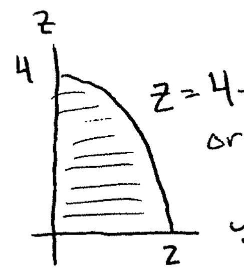


$z=4-y^2$





look down x-axis



$z=4-y^2$
or $y^2=4-z$
or $y=\sqrt{4-z}$

$0 \le z \le 4$
$0 \le y \le \sqrt{4-z}$

$$\boxed{\int_0^4 \int_0^{\sqrt{4-z}} \int_{\frac{1}{2}y}^{1} f(x,y,z)\, dx\, dy\, dz}$$

For x direction move between surfaces

$y=2x$ or $x=\frac{1}{2}y$ (back)
$x=1$ (front)

score

3. Find the work done by the force field $\mathbf{F} = (x+y)\,\mathbf{i} + xy\,\mathbf{j}$ in moving a particle along the curve C given by $x = 2t$, $y = t^2 - 1$, $0 \leqslant t \leqslant 2$.

$$W = \int_C F \cdot dr = \int_C (x+y)\,dx + (xy)\,dy$$

$$= \int_0^2 \left[(2t + (t^2-1)) \cdot 2 + (2t)(t^2-1)(2t) \right] dt$$

$$= \int_0^2 \left(4t + 2t^2 - 2 + 4t^4 - 4t^2 \right) dt$$

$$= \int_0^2 \left(4t^4 - 2t^2 + 4t - 2 \right) dt$$

$$= \left(\frac{4}{5}t^5 - \frac{2}{3}t^3 + 2t^2 - 2t \right) \Big|_0^2$$

$$= \left[\frac{4}{5} \cdot 32 - \frac{2}{3} \cdot 8 + 2 \cdot 2^2 - 2 \cdot 2 \right) - 0$$

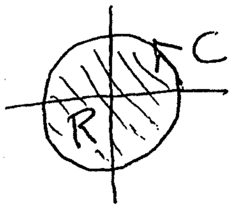
$$= \frac{128}{5} - \frac{16}{3} + 4$$

$$= \frac{384 - 80 + 60}{15}$$

$$= \boxed{\frac{364}{15}}$$

score

2. Evaluate $\oint_C \overbrace{(3y - \arctan(x^6))}^{M}\,dx + \overbrace{(7x - \cos(\sqrt{y-1}))}^{N}\,dy$ where C is the circle $x^2 + y^2 = 1$ traversed counterclockwise.



Use Green's Theorem (closed curve in plane)

$$= \iint_R (N_x - M_y)\,dA$$

$$= \iint_R (7 - 3)\,dA$$

$$= 4 \iint_R dA$$

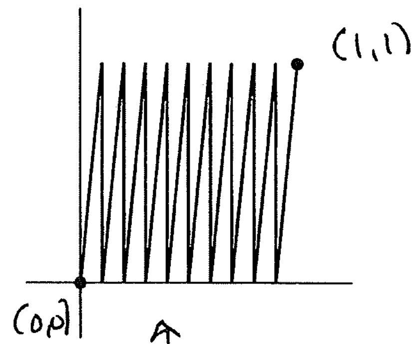
$$= 4\,(\text{Area of } R)$$ circle of radius 1 has area $\pi \cdot 1^2$

$$= \boxed{4\pi}$$

score

5. A sawtooth curve C (see below) consists of 19 line segments *starting* at $(0,0)$ and *ending* at $(1,1)$. Find the value b, given that

$$\int_C \underbrace{\left(-\pi\sin(\pi x)+4e^{1-x}\arctan(y)\right)}_{M=f_x} dx + \underbrace{\left(by-\frac{4e^{1-x}}{1+y^2}\right)}_{N=f_y} dy = 5-\pi.$$



unpleasant curve, so check for conservative

$$f = \cos(\pi x) - 4e^{1-x}\arctan(y) + \tfrac{1}{2}by^2$$

check

$$f_x = -\pi\sin(\pi x) + 4e^{1-x}\arctan(y)$$

$$f_y = -4e^{1-x}\cdot\frac{1}{1+y^2} + by$$

$$5-\pi = \left(\cos(\pi x) - 4e^{1-x}\arctan(y) + \tfrac{1}{2}by^2\right)\Big|_{(0,0)}^{(1,1)}$$

$$= \Big(-1-4e^0\underbrace{\arctan(1)}_{\pi/4}+\tfrac{1}{2}b\Big) - \Big(1-4\underbrace{e^1\arctan(0)}_{=0}+0\Big)$$

$$= -2-\pi+\tfrac{1}{2}b$$

$$\rightsquigarrow \tfrac{1}{2}b = 7$$

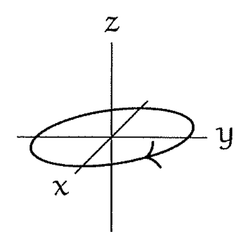
$$\rightsquigarrow b = \boxed{14}$$

score

1. Evaluate $\displaystyle\int_{-1}^{0}\int_{0}^{1} \cos(y^3+y)\,dy\,dx + \int_{0}^{3}\int_{\sqrt{x/3}}^{1} \cos(y^3+y)\,dy\,dx.$

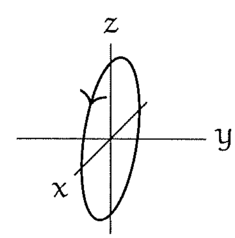
1. Evaluate $\displaystyle\int_{1}^{e^2}\int_{\ln y}^{2} e^{(e^x-x)}\,dx\,dy.$

1. Evaluate $\displaystyle\int_{0}^{4}\int_{y/4}^{\sqrt[3]{y/4}} \sin(2x^2-x^4)\,dx\,dy.$



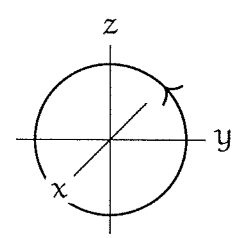
2. Let C be the parametric curve $(2\sin t, 2\cos t, 0)$ with $0 \leqslant t \leqslant 2\pi$ (a circle of radius 2 in the xy-plane; see figure to the right where the orientation is also indicated). Find

$$\int_C \langle \cos(x^3)+y, e^y, x^2y\rangle \cdot \mathbf{T}\,ds.$$



2. Let C be the parametric curve $(2\sin t, 0, 2\cos t)$ with $0 \leqslant t \leqslant 2\pi$ (a circle of radius 2 in the xz-plane; see figure to the right where the orientation is also indicated). Find

$$\int_C \langle e^x+2z, x^3z^2, \ln(1+z^2)\rangle \cdot \mathbf{T}\,ds.$$

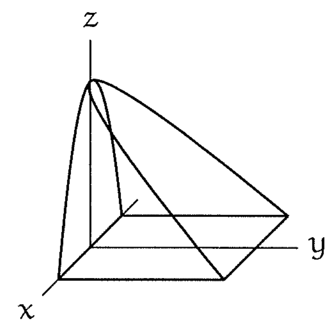


2. Let C be the parametric curve $(0, 2\cos t, 2\sin t)$ with $0 \leqslant t \leqslant 2\pi$ (a circle of radius 2 in the yz-plane; see figure to the right where the orientation is also indicated). Find

$$\int_C \langle y^2+z^2, \sin(y^2+y)-3z, \arctan(z^4)\rangle \cdot \mathbf{T}\,ds.$$

5. Let S be the solid $-2 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 4$, $0 \leqslant z \leqslant 4 - x^2$ and $y + z \leqslant 4$ (see picture to the right).

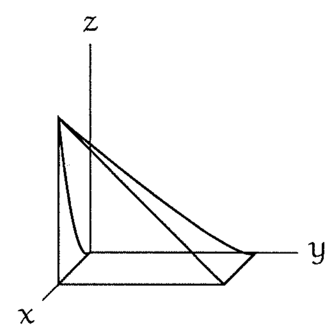
   Set the bounds for the following integrals over S.



$$\int_{\_\_}^{\_\_}\int_{\_\_\_\_}^{\_\_\_\_}\int_{\_\_\_\_\_}^{\_\_\_\_\_} f(x,y,z)\,dy\,dz\,dx \quad \text{and} \quad \int_{\_\_}^{\_\_}\int_{\_\_\_\_}^{\_\_\_\_}\int_{\_\_\_\_\_}^{\_\_\_\_\_} f(x,y,z)\,dx\,dy\,dz$$

5. Let S be the solid $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 4$, $0 \leqslant z \leqslant x^2$ and $y + z \leqslant 4$ (see picture to the right).

   Set the bounds for the following integrals over S.



$$\int_{\_\_}^{\_\_}\int_{\_\_\_\_}^{\_\_\_\_}\int_{\_\_\_\_\_}^{\_\_\_\_\_} f(x,y,z)\,dy\,dz\,dx \quad \text{and} \quad \int_{\_\_}^{\_\_}\int_{\_\_\_\_}^{\_\_\_\_}\int_{\_\_\_\_\_}^{\_\_\_\_\_} f(x,y,z)\,dx\,dy\,dz$$

4. You are hiking in the "3-D Forest", where your elevation above sea level z, measured in cubits, is given by a differentiable function $f(x, y)$ – where (x, y) represents a sea-level grid position measured in furlongs.
You stop at the point $(3, 2, 20)$ and notice that you are standing in the middle of a small stream. From that point, the stream flows in the direction $\langle -3, 4\rangle$ where the elevation is dropping at a rate of 15 cubits/furlong.
opposite gradient
Since your feet are beginning to get cold, you step out of the stream in the direction $\langle -2, -2\rangle$. As you take that step, are you climbing or descending? At what rate (include units)?

We can find the gradient

$$\nabla F = -15 \frac{\langle -3,4\rangle}{|\langle -3,4\rangle|} = \frac{-15\langle -3,4\rangle}{\sqrt{9+16}} = -3\langle -3,4\rangle = \langle 9,-12\rangle$$

now find directional derivative

$$\nabla f \cdot \frac{\langle -2,-2\rangle}{|\langle -2,-2\rangle|} = \frac{\langle 9,-12\rangle \cdot \langle -2,-2\rangle}{\sqrt{4+4}} = \frac{-18+24}{2\sqrt{2}} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}}$$

positive directional derivative, so climbing at a rate of $\boxed{\frac{3}{\sqrt{2}}\ \frac{\text{cubits}}{\text{Furlong}}}$

score

Closed oriented curve in plane $\longleftrightarrow$ Green's Theorem

4. Evaluate

$$\oint_C \underbrace{\left(\sin\left(x^2+3x\right)+5y-e^{\cos x}\right)}_{M}\,dx+\underbrace{\left(2x+\ln\left(y^2+1\right)\right)}_{N}\,dy$$

where C is the path composed of line segments in the xy-plane connecting $(3,0)$ to $(0,5)$ to $(-1,0)$ to $(2,-1)$ back to $(3,0)$.

$$\oint_C M\,dx+N\,dy=\iint_R (N_x-M_y)\,dA$$

$$=\iint_R (2-5)\,dA$$

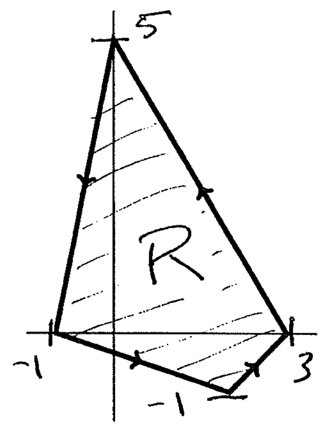
$$=-3\iint_R dA$$

$$=-3\,(\text{Area of } R)$$

$$=-3\left(\tfrac{1}{2}\cdot 4\cdot 5+\tfrac{1}{2}\cdot 4\cdot 1\right)$$

$$=-3\,(10+2)$$

$$=\boxed{-36}$$



area can be found by two triangles (above x-axis and below x-axis)

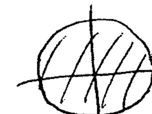
score

2. Convert **DO NOT EVALUATE** the following integral

$$\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_{1}^{\sqrt{4-x^2-y^2}} (x^2+y^2)\,dz\,dx\,dy$$

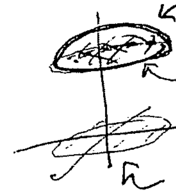
$-\sqrt{3} \le y \le \sqrt{3}$

$\underbrace{-\sqrt{3-y^2} \le x \le \sqrt{3-y^2}}$



circle of radius $\sqrt{3}$ in the plane

$\underset{\text{plane}}{1} \le z \le \underbrace{\sqrt{4-x^2-y^2}}_{\text{sphere}}$



(a) into cylindrical coordinates;

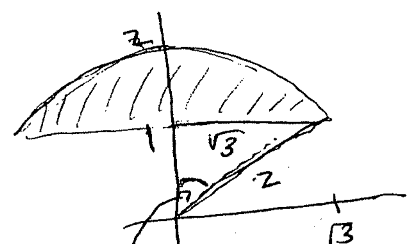
$$\boxed{\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r^2\cdot r\,dz\,dr\,d\theta}$$

$z=1 \longleftrightarrow z=1$

$z=\sqrt{4-x^2-y^2} \longleftrightarrow z=\sqrt{4-r^2}$

(b) into spherical coordinates.

look at slice



angle is $\frac{\pi}{3}$

$$\boxed{\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2\sin^2\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

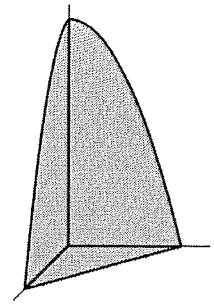
$z=1 \longleftrightarrow \rho\cos\phi=1$, or $\rho=\frac{1}{\cos\phi}=\sec\phi$

$z=\sqrt{4-x^2-y^2}$ or $x^2+y^2+z^2=4$ $\longleftrightarrow \rho^2=4$ or $\rho=2$

$x^2+y^2=r^2=\rho^2\sin^2\phi$

score

Solutions to practice problems for Exam 3 (MATH 265; Fall 17)



1. Let S be the solid in the positive orthant ($x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$) bounded by the surface $z = 4-(x+y)^2$ (see picture to the right). Set up the bounds for the following two integrals over S:

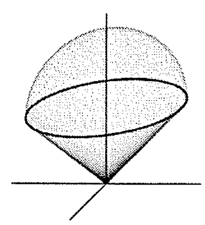
$$\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x,y,z)\,dx\,dy\,dz \quad\text{and}\quad \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} f(x,y,z)\,dz\,dy\,dx$$

Note if we look down the x-axis we see the parabola $z = 4-y^2$ (or $y = \sqrt{4-z}$) in the yz-plane which helps us to determine the outer two sets of bounds. For the inner set we solve the surface for x and get $x = \sqrt{4-z}-y$. So we have

$$\boxed{\int_0^4\int_0^{\sqrt{4-z}}\int_0^{\sqrt{4-z}-y} f(x,y,z)\,dx\,dy\,dz}\,.$$

If we look down the z-axis we see a triangle with vertices at $(0,0)$, $(2,0)$ and $(0,2)$ in the xy-plane which helps us determine the outer two sets of bounds. So we have

$$\boxed{\int_0^2\int_0^{2-x}\int_0^{4-(x+y)^2} f(x,y,z)\,dz\,dy\,dx}\,.$$



2. Set up (but do not evaluate) an integral in spherical coordinates to find the mass of the object which lies above the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+(z-1)^2 = 1$ (see picture to the right) given the density function $\delta(x,y,z) = \sqrt{x^2+y^2}$.

Because of symmetry we will have $0 \leqslant \theta \leqslant 2\pi$. To find mass we integrate density which we note can be written as $\delta = r = \rho\sin\phi$. The sphere can be rewritten as $x^2+y^2+z^2-2z+1 = 1$ or $x^2+y^2+z^2 = 2z$ or $\rho^2 = 2\rho\cos\phi$ or $\rho = 2\cos\phi$ (this is our upper limit to where ρ goes). Finally note that the cone corresponds to $\phi = \frac{1}{4}\pi$ and that we are interested in $0 \leqslant \phi \leqslant \frac{1}{4}\pi$. So putting this altogether we have

$$M = \iiint_S \delta\,dV = \underbrace{\int_0^{2\pi}\int_0^{\frac{1}{4}\pi}\int_0^{2\cos\phi}}_{\text{bounds}} \underbrace{\rho\sin\phi}_{=\delta}\,\underbrace{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}_{=dV} = \boxed{\int_0^{2\pi}\int_0^{\frac{1}{4}\pi}\int_0^{2\cos\phi} \rho^3\sin^2\phi\,d\rho,\,d\phi,\,d\theta}\,.$$