3. Find a_T and a_N (the tangential and normal components of acceleration) for the curve $\mathbf{r}(t) = \langle \frac{2}{3}t^3, t, t^2 \rangle$. Simplify your answer as much as possible.

We have

- $\mathbf{r}'(t) = \langle 2t^2, 1, 2t \rangle$ and $\mathbf{r}''(t) = \langle 4t, 0, 2 \rangle$.
- $|\mathbf{r}'(t)| = \sqrt{4t^4 + 1 + 4t^2} = \sqrt{(2t^2+1)^2} = 2t^2 + 1$
- $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \langle 2t^2, 1, 2t \rangle \cdot \langle 4t, 0, 2 \rangle = 8t^3 + 0 + 4t = 4t(2t^2+1)$
- $\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 2t^2, 1, 2t \rangle \times \langle 4t, 0, 2 \rangle = \langle 2, 4t^2, -4t \rangle$
- $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{4 + 16t^4 + 16t^2} = \sqrt{4(4t^4 + 4t^2 + 1)} = \sqrt{4(2t^2+1)^2} = 2(2t^2+1)$
- $a_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4t(2t^2+1)}{2t^2+1} = 4t$

  $a_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{2(2t^2+1)}{2t^2+1} = 2$

Note that once a_T has been computed the value a_N can be computed without using a cross product. Namely since $|\mathbf{a}|^2 = a_T^2 + a_N^2$ and $a_N \geqslant 0$ we have
$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{((4t)^2 + (2)^2) - (4t)^2} = \sqrt{4} = 2.$

---

3. Find κ (curvature) for the curve $\mathbf{r}(t) = \langle \frac{2}{3}t^3, t, t^2 \rangle$. Simplify your answer as much as possible.

We have

- $\mathbf{r}'(t) = \langle 2t^2, 1, 2t \rangle$ and $\mathbf{r}''(t) = \langle 4t, 0, 2 \rangle$.
- $|\mathbf{r}'(t)| = \sqrt{4t^4 + 1 + 4t^2} = \sqrt{(2t^2+1)^2} = 2t^2 + 1$
- $\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 2t^2, 1, 2t \rangle \times \langle 4t, 0, 2 \rangle = \langle 2, 4t^2, -4t \rangle$
- $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{4 + 16t^4 + 16t^2} = \sqrt{4(4t^4 + 4t^2 + 1)} = \sqrt{4(2t^2+1)^2} = 2(2t^2+1)$
- $\kappa = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{2(2t^2+1)}{(2t^2+1)^3} = \dfrac{2}{(2t^2+1)^2}$

4. Find a_T and a_N (the tangential and normal components of acceleration) for the curve $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 2\mathbf{k}$. Simplify your answer as much as possible.

Note that

$$\begin{aligned}
\mathbf{r}'(t) &= (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j} + 0\mathbf{k} = t\cos t\mathbf{i} + t\sin t\mathbf{j}, \\
\mathbf{r}''(t) &= (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}.
\end{aligned}$$

And further note

$$\begin{aligned}
|\mathbf{r}'(t)| &= \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2(\cos^2 t + \sin^2 t)} = t \\
|\mathbf{r}''(t)| &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} \\
&= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t} \\
&= \sqrt{(\cos^2 + \sin^2 t) + t^2(\sin^2 t + \cos^2 t)} = \sqrt{1+t^2}
\end{aligned}$$

So we have $a_T = \frac{d}{dt}(|\mathbf{r}'(t)|) = \frac{d}{dt}(t) = 1$. And moreover we note that $1 + a_N^2 = a_T^2 + a_N^2 = |\mathbf{r}''(t)|^2 = 1 + t^2$ allowing us to conclude that $a_N^2 = t^2$ and so $a_N = t$. (Technically there is a subtle sign reversal for $t < 0$; but not the key part of this problem.)

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4. Find the curvature κ for the curve $\mathbf{r}(t) = \langle 3t, 4\sin t, 4\cos t\rangle$ at any time t. Simplify your answer as much as possible.

We have

$$\begin{aligned}
\mathbf{r}'(t) &= \langle 3, 4\cos t, -4\sin t\rangle \\
\mathbf{r}''(t) &= \langle 0, -4\sin t, -4\cos t\rangle.
\end{aligned}$$

We note that

$$|\mathbf{r}'(t)| = \sqrt{(3)^2 + (4\cos t)^2 + (-4\sin t)^2} = \sqrt{9+16} = 5.$$

Next we have

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 3, 4\cos t, -4\sin t\rangle \times \langle 0, -4\sin t, -4\cos t\rangle = \langle -16, 12\cos t, -12\sin t\rangle$$

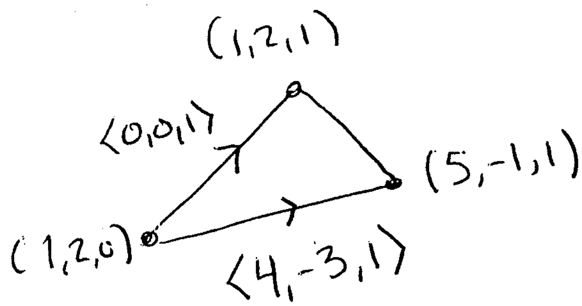
so

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle -16, 12\cos t, -12\sin t\rangle| = \sqrt{256 + 144\cos^2 t + 144\sin^2 t} = \sqrt{400} = 20.$$

Putting it altogether we have

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{20}{5^3} = \frac{4}{25}$$

1. Find the area of a triangle formed by the three points $(1,2,0)$, $(1,2,1)$, $(5,-1,1)$.



$$\text{Area} = \frac{1}{2}\left|\langle 0,0,1\rangle \times \langle 4,-3,1\rangle\right|$$

$$= \frac{1}{2}\left|\langle 3,4,0\rangle\right|$$

$$= \frac{1}{2}\sqrt{9+16+0}$$

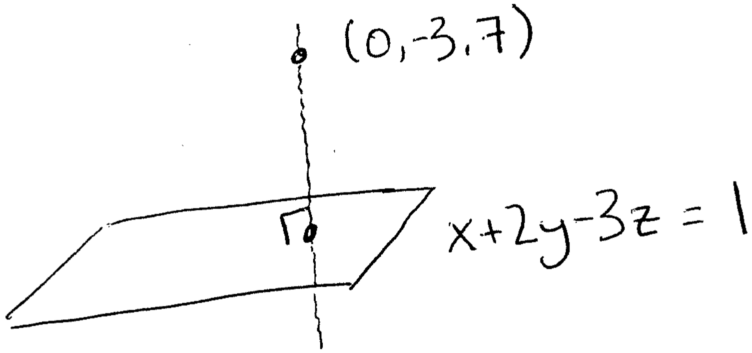
$$= \frac{1}{2}\sqrt{25} = \boxed{\frac{5}{2}}$$

$$\langle 0,0,1\rangle \times \langle 4,-3,1\rangle = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 4 & -3 & 1 \end{vmatrix} = \begin{matrix} 0i + 4j + 0k \\ +3i - 0j - 0k \end{matrix}$$

$$= \langle 3,4,0\rangle$$

score

2. Find the point in the plane $x + 2y - 3z = 1$ which is closest to the point $P = (0, -3, 7)$, *and* find the shortest distance from P to the plane.



Find line with point $(0,-3,7)$ and direction $\langle 1,2,-3 \rangle$

$$\left.\begin{aligned} x &= 0 + t \\ y &= -3 + 2t \\ z &= 7 - 3t \end{aligned}\right]$$ find intersection of line with the plane.

$$t + 2(-3+2t) - 3(7-3t) = 1$$

$$14t - 27 = 1$$

$$14t = 28$$

$$t = 2 \leftarrow \text{put into line to get point}$$

$$\boxed{(2,1,1)}$$

Distance to plane is distance between $(0,-3,7)$ and $(2,1,1)$

$$= \sqrt{(2-0)^2 + (1-(-3))^2 + (1-7)^2} = \sqrt{4+16+36} = \boxed{\sqrt{56}}$$

---

alternative

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}} = \frac{|0-6-21-1|}{\sqrt{1^2+2^2+(-3)^2}} = \frac{28}{\sqrt{14}} \cdot \frac{\sqrt{14}}{\sqrt{14}} = \boxed{2\sqrt{14}}$$

score

3. A particle is traveling through space with acceleration $\mathbf{a}(t) = \langle e^t, -e^{-t}, 4e^{2t} \rangle$ at time t. At time $t = 0$, the particle is at the initial point $(3,1,2)$ and has initial velocity $\langle -1,4,0 \rangle$. Find the position of the particle at time $t = 1$.

$$a(t) = \langle e^t, -e^{-t}, 4e^{2t} \rangle$$

$$v(t) = \langle e^t + C, e^{-t} + D, 2e^{2t} + F \rangle$$

$$v(0) = \langle -1, 4, 0 \rangle = \langle 1 + C, 1 + D, 2 + E \rangle$$

$$C = -2, \; D = 3, \; E = -2$$

$$v(t) = \langle e^t - 2, e^{-t} + 3, 2e^{2t} - 2 \rangle$$

$$r(t) = \langle e^t - 2t + F, -e^{-t} + 3t + G, e^{2t} - 2t + H \rangle$$

$$r(0) = \langle 3, 1, 2 \rangle = \langle 1 + F, -1 + G, 1 + H \rangle$$

$$F = 2, \; G = 2, \; H = 1$$

$$r(t) = \langle e^t - 2t + 2, -e^{-t} + 3t + 2, e^{2t} - 2t + 1 \rangle$$

Position at time 1:

$$r(1) = \langle e - 2 + 2, -e^{-1} + 3 + 2, e^2 - 2 + 1 \rangle$$

$$= \boxed{\langle e, -\tfrac{1}{e} + 5, e^2 - 1 \rangle}$$

score

4. Consider the curve $\mathbf{r}(t) = \langle t^2, t - \frac{t^3}{3}, t + \frac{t^3}{3} + 1 \rangle$, where $0 \leqslant t < \infty$.

(a) Find the cumulative arc length function $s(t)$ starting from $t_0 = 0$.

$\hookrightarrow s(t) = \int_0^t |r'(\tau)| \, d\tau$

$r'(t) = \langle 2t, 1 - t^2, 1 + t^2 \rangle$

$$|r'(t)| = \sqrt{(2t)^2 + (1-t^2)^2 + (1+t^2)^2}$$
$$= \sqrt{4t^2 + 1 - 2t^2 + t^4 + 1 + 2t^2 + t^4}$$
$$= \sqrt{2t^4 + 4t^2 + 2}$$
$$= \sqrt{2(t^4 + 2t^2 + 1)}$$
$$= \sqrt{2(t^2+1)^2}$$
$$= \sqrt{2}\,(t^2+1)$$

$$s(t) = \int_0^t \sqrt{2}(\tau^2+1)\,d\tau = \sqrt{2}\left(\tfrac{1}{3}\tau^3 + \tau\right)\Big|_0^t = \boxed{\sqrt{2}\left(\tfrac{1}{3}t^3 + t\right)}$$

(b) Find the length of the curve from $\underbrace{(0,0,1)}_{\text{time } t=0}$ to $\underbrace{(9,-6,13)}_{\text{time } t=3}$.

length from time $0 \leq t \leq 3$ $= s(3) = \sqrt{2}\left(\frac{1}{3} \cdot 3^3 + 3\right) = \boxed{12\sqrt{2}}$

score

5. Consider the following two lines.

$$\begin{array}{l|l} x = 1 + 2t & x = 5 + 2s \\ y = 2t & y = 1 - s \\ z = -2 + 3t & z = 2 + s \end{array}$$

(a) Show these two lines intersect by finding the *point* of intersection, and verifying this is a point on *both* lines.

$$\left.\begin{array}{l} 1+2t = 5+2s \\ 2t = 1-s \end{array}\right] \rightarrow \begin{array}{r} -2s+2t = 4 \\ -(s+2t = 1) \\ \hline -3s = 3 \end{array}$$

$s = -1 \rightsquigarrow 2t = 1-(-1)$
$t = 1$

So intersection would be at $s=-1$ and $t=1$

check

$$\left.\begin{array}{l} (1+2\cdot 1, 2\cdot 1, -2+3\cdot 1) = (3,2,1) \\ (5+2(-1), 1-(-1), 2+(-1)) = (3,2,1) \end{array}\right\}$$

point $\boxed{(3,2,1)}$ is on both lines.

(b) Find the equation of the plane containing *both* lines.

normal vector from cross product of direction vectors

$$\langle 2,2,3\rangle \times \langle 2,-1,1\rangle = \begin{vmatrix} i & j & k \\ 2 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{array}{l} 2i + 6j - 2k \\ +3i - 2j - 4k \end{array}$$

$$= \langle 5, 4, -6\rangle$$

For point we can use answer from (a) or any point on either line (for example $(1,0,-2)$ or $(5,1,2)$).

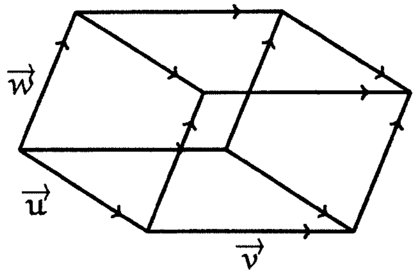
$$\boxed{5x + 4y - 6z = 17}$$

$$\underbrace{5\cdot 3}_{15} + \underbrace{4\cdot 2}_{8} - \underbrace{6\cdot 1}_{6}$$

score

1. Find the total *surface area* (**not** volume) from all *six sides* of the parallelepiped with the vectors corresponding to the edges being: $\vec{u} = \langle 3,1,0\rangle$, $\vec{v} = \langle 3,1,\sqrt{10}\rangle$, and $\vec{w} = \langle -1,0,0\rangle$.

Sides consist of six parallelograms, use cross products to find area.



$$\text{Area} = 2|\vec{u}\times\vec{v}| + 2|\vec{u}\times\vec{w}| + 2|\vec{v}\times\vec{w}|$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ 3 & 1 & \sqrt{10} \end{vmatrix} = \begin{matrix} \sqrt{10}\,i + 0j + 3k \\ -0i + 3\sqrt{10}\,j - 3k \end{matrix} = \langle \sqrt{10}, 3\sqrt{10}, 0\rangle$$

$$|\vec{u}\times\vec{v}| = \sqrt{10+90+0} = \sqrt{100} = 10$$

$$\vec{u}\times\vec{w} = \begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = \begin{matrix} 0i + 0j + 0k \\ -0i - 0j + k \end{matrix} = \langle 0,0,1\rangle$$

$$|\vec{u}\times\vec{w}| = \sqrt{0+0+1} = \sqrt{1} = 1$$

$$\vec{v}\times\vec{w} = \begin{vmatrix} i & j & k \\ 3 & 1 & \sqrt{10} \\ -1 & 0 & 0 \end{vmatrix} = \begin{matrix} 0i - \sqrt{10}\,j + 0k \\ -0i - 0j + k \end{matrix} = \langle 0, -\sqrt{10}, 1\rangle$$

$$|\vec{v}\times\vec{w}| = \sqrt{0+10+1} = \sqrt{11}$$

$$\text{Area} = 2\cdot 10 + 2\cdot 1 + 2\cdot\sqrt{11} = \boxed{22 + 2\sqrt{11}}$$

score

2. Find the distance traveled (i.e., arc length) between times $t = 0$ and $t = 1$ along the curve $r(t) = \langle 2t^2, 7 - \frac{8}{3}t^{3/2}, \frac{2}{5}t^{5/2} - 2t^{3/2} + 1\rangle$. Simplify your answer as much as possible.

$$r'(t) = \langle 4t, -4t^{1/2}, t^{3/2} - 3t^{1/2}\rangle$$

$$|r'(t)| = \sqrt{(4t)^2 + (-4t^{1/2})^2 + (t^{3/2} - 3t^{1/2})^2}$$

$$= \sqrt{16t^2 + 16t + t^3 - 6t^2 + 9t}$$

$$= \sqrt{t^3 + 10t^2 + 25t}$$

$$= \sqrt{t(t^2 + 10t + 25)}$$

$$= \sqrt{t(t+5)^2}$$

$$= \sqrt{t}\,(t+5)$$

$$= t^{3/2} + 5t^{1/2}$$

$$\text{Distance} = \int_0^1 |r'(t)|\,dt = \int_0^1 (t^{3/2} + 5t^{1/2})\,dt$$

$$= \left(\frac{2}{5}t^{5/2} + \frac{10}{3}t^{3/2}\right)\Big|_0^1$$

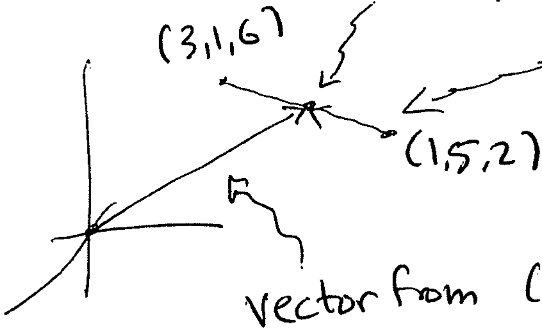
$$= \left(\frac{2}{5} + \frac{10}{3}\right) - 0$$

$$= \frac{6 + 50}{15} = \boxed{\frac{56}{15}}$$

score

3. A line passes through the origin and through the midpoint between $A = (3, 1, 6)$ and $B = (1, 5, 2)$. Find $\cos\theta$ where θ is the angle between the line and the line segment connecting A and B.

midpoint is $(2,3,4)$



vector from $(1,5,2)$ to $(2,3,4)$
is $\langle 1,-2,2\rangle$

vector from $(0,0,0)$ to $(2,3,4)$
is $\langle 2,3,4\rangle$

need angle between $u = \langle 1,-2,2\rangle$ and $v = \langle 2,3,4\rangle$

$$\cos\theta = \frac{u \cdot v}{|u|\,|v|} = \frac{\langle 1,-2,2\rangle \cdot \langle 2,3,4\rangle}{|\langle 1,-2,2\rangle|\,|\langle 2,3,4\rangle|}$$

$$= \frac{2-6+8}{\sqrt{1+4+4}\,\sqrt{4+9+16}}$$

$$= \boxed{\frac{4}{3\sqrt{29}}}$$

score

4. Find the components of acceleration, namely a_T and a_N, for the vector valued function $\mathbf{r}(t) = (e^t\cos(t))\mathbf{i} + (e^t\sin(t))\mathbf{j} + t\mathbf{k}$ at time $t = 0$.

$$r'(t) = (e^t\cos t - e^t \sin t)i + (e^t \sin t + e^t \cos t)j + k$$

$$|r'(t)| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + 1}$$

$$= \sqrt{\underbrace{e^{2t}\cos^2 t \cancel{-2e^{2t}\cos t\sin t} + e^{2t}\sin^2 t}_{e^{2t}} + \underbrace{e^{2t}\sin^2 t \cancel{+2e^{2t}\sin t\cos t} + e^{2t}\cos^2 t}_{e^{2t}} + \underbrace{1}_{1}}$$

$$= \sqrt{2e^{2t}+1} = (2e^{2t}+1)^{1/2}$$

$$\frac{d}{dt}(|r'(t)|) = \frac{1}{2}(2e^{2t}+1)^{-1/2}(4e^{2t})$$

$$a_T = \frac{d}{dt}(|r'(t)|)\Big|_{t=0} = \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\cdot 4 = \boxed{\frac{2}{3}\sqrt{3} = a_T}$$

$$r''(t) = (e^t\cancel{\cos t} - e^t\sin t - e^t\sin t \cancel{- e^t\cos t})i + (\cancel{e^t\sin t} + e^t\cos t + e^t\cos t \cancel{- e^t\sin t})j$$

$$= (-2e^t\sin t)i + (2e^t\cos t)j$$

$$r''(0) = 2j$$

$$\underbrace{a_T^2}_{\frac{4}{3}} + a_N^2 = \underbrace{|r''(0)|}_{4} \rightsquigarrow a_N^2 = 4 - \frac{4}{3} = \frac{8}{3}$$

$$\boxed{a_N = \frac{2}{3}\sqrt{6}}$$

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alternate $r'(0) = \langle 1,1,1\rangle$ $\quad |r'(0)| = \sqrt{3}$

$r''(0) = \langle 0,2,0\rangle$

$$r'(0)\cdot r''(0) = \langle 1,1,1\rangle \cdot \langle 0,2,0\rangle = 2$$

$$r'(0)\times r''(0) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 0\end{vmatrix} = \begin{matrix} 0i + 0j + 2k \\ -2i - 0j - 0k\end{matrix} = \langle -2, 0, 2\rangle$$

$$|r'(0)\times r''(0)| = \sqrt{4+0+4} = \sqrt{8} = 2\sqrt{2}$$

$$\boxed{\begin{aligned} a_T &= \frac{r'\cdot r''}{|r'|} = \frac{2}{\sqrt{3}} \\ a_N &= \frac{|r'\times r''|}{|r'|} = \frac{\sqrt{8}}{\sqrt{3}}\end{aligned}}$$

score