MATH 2630 TEST 2 PRACTICE PROBS + SOLNS

1. Let $f(x, y) = 2x^2 - 3xy + y^2 + y$, find the directional derivative of f at $(2, -1)$ in the direction of the the vector $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

↳ make into a unit vector $|v| = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$$u = \frac{1}{|v|} v = \frac{4}{5} i - \frac{3}{5} j = \left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle$$

$$\nabla f(x,y) = \langle 4x - 3y, -3x + 2y + 1 \rangle$$

$$\nabla f(2,-1) = \langle 8+3, -6-2+1 \rangle$$

$$= \langle 11, -7 \rangle$$

$$D_u f(p) = \nabla f(p) \cdot u$$

$$= \langle 11, -7 \rangle \cdot \left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle$$

$$= \frac{44}{5} + \frac{21}{5}$$

$$= \frac{65}{5}$$

$$= \boxed{13}$$

score

2. Find the tangent plane to the surface $z + 1 = xe^{3y}\cos(7z)$ at $(1, 0, 0)$

$$\underbrace{xe^{3y}\cos(7z) - z}_{F(x,y,z)} = 1$$

$$\nabla F = \langle e^{3y}\cos(7z), 3xe^{3y}\cos(7z), -7xe^{3y}\sin(7z) - 1\rangle$$

$$\nabla F(1,0,0) = \langle e^{0}\cos(0), 3\cdot 1\cdot e^{0}\cos(0), -7\cdot 1\cdot e^{0}\cdot\sin(0) - 1\rangle$$

$$= \underbrace{\langle 1, 3, -1\rangle}_{\text{normal vector for tangent plane}}$$

$$x + 3y - z = d$$

plug in $(1,0,0)$

$$1 + 0 - 0 = d$$

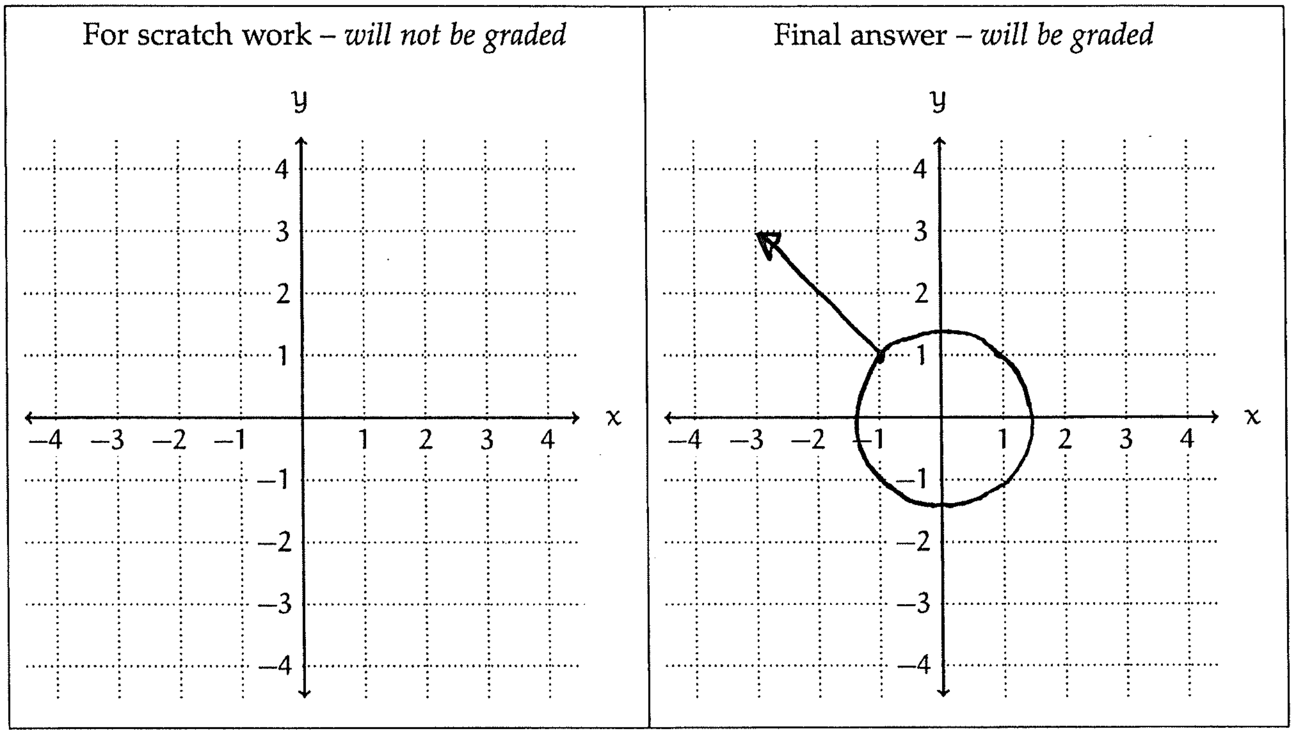
$$\boxed{x + 3y - z = 1}$$

score

3. Consider the paraboloid $f(x, y) = x^2 + y^2$.

$x^2 + y^2 = 2$ ← circle
center = $(0,0)$
radius = $\sqrt{2}$

(a) Sketch the level curve corresponding with $f(x, y) = 2$ in the grid below.



(b) Compute $\nabla f(-1, 1)$, and draw this vector in the grid above where the starting point of the vector is at the point $(-1, 1)$.

$\nabla f(x,y) = \langle 2x, 2y \rangle$

$\nabla f(-1,1) = \langle -2, 2 \rangle$

score

4. Use the linearization of the function

$$f(x,y) = \sqrt{2x+3y} - \frac{x}{y} \quad = (2x+3y)^{1/2} - \frac{x}{y}$$

at the point $(3,1)$ to estimate the value of $f(2.97, 1.02)$.

$$L(x,y) = f(3,1) + f_x(3,1)(x-3) + f_y(3,1)(y-1)$$

$$= 0 - \frac{2}{3}(x-3) + \frac{7}{2}(y-1)$$

$$f(2.97,1.02) \approx L(2.97, 1.02) = -\frac{2}{3}(2.97-3) + \frac{7}{2}(1.02-1)$$

$$= -\frac{2}{3}(-.03) + \frac{7}{2}(.02)$$

$$= .02 + .07$$

$$= \boxed{.09}$$

$$f(x,y) = (2x+3y)^{1/2} - \frac{x}{y}$$

$$f_x(x,y) = \frac{1}{2}(2x+3y)^{-1/2} \cdot 2 - \frac{1}{y}$$

$$f_y(x,y) = \frac{1}{2}(2x+3y)^{-1/2} \cdot 3 + \frac{x}{y^2}$$

$$f(3,1) = (6+3)^{1/2} - \frac{3}{1} = 3-3 = 0$$

$$f_x(3,1) = \frac{1}{2}(6+3)^{-1/2} 2 - 1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

$$f_y(3,1) = \frac{1}{2}(6+3)^{-1/2} \cdot 3 + \frac{3}{1^2} = \frac{1}{2} + 3 = \frac{7}{2}$$

Note $f(2.97, 1.02) = 0.088235\ldots$

so a reasonable approximation

score

5. Find and classify the critical points of $h(x, y) = \frac{1}{3}x^3 - x^2y + 18y^2 - 72y - 24$. (Hint: when determining the sign of an expression, pull out common factors to simplify.)

$h_x = x^2 - 2xy = 0 \rightsquigarrow x(x-2y) = 0$ so $x = 0$ OR $x = 2y$

$h_y = -x^2 + 36y - 72 = 0$

$\xrightarrow{x=0} 36y - 72 = 0$ or $y = \frac{72}{36} = 2$

giving point $(0,2)$

$\xrightarrow{x=2y} -(2y)^2 + 36y - 72 = 0$

$-4y^2 + 36y - 72 = 0$

$y^2 - 9y + 18 = 0$

$(y-3)(y-6) = 0$ or $y = 3, 6$

giving points $(6,3)$, $(12,6)$

$D = h_{xx}h_{yy} - (h_{xy})^2 = (2x-2y)(36) - (-2x)^2$

$D(0,2) = (-4)(36) - 0 < 0 \rightsquigarrow$ SADDLE

$D(6,3) = (12-6)(36) - (-12)^2 = 6\cdot 36 - 12\cdot 12 = 6\cdot 12\underbrace{(3-2)}_{>0} > 0$

and $f_{yy} = 36 > 0 \rightsquigarrow$ LOCAL MIN

$D(12,6) = (24-12)(36) - (-24)^2 = 12\cdot 36 - 24\cdot 24 = 12\cdot 12\underbrace{(3-2\cdot 2)}_{<0} < 0$

$\rightsquigarrow$ SADDLE

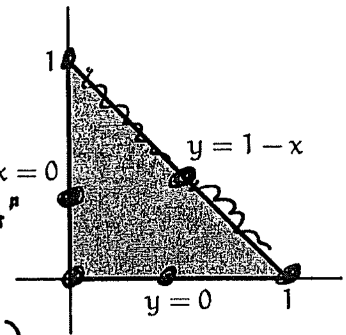
$\boxed{\begin{array}{ll} (0,2) & \text{SADDLE} \\ (6,3) & \text{LOCAL MIN} \\ (12,6) & \text{SADDLE} \end{array}}$

score

6. Find the absolute (global) maximum and minimum values of

$$f(x,y) = y^2 - x^2 - y + x$$

in the closed triangular region bounded by the lines $x = 0$, $y = 0$ and $y = 1 - x$.



Interior $\quad \nabla f = \langle 2y-1, -2x+1 \rangle$

$$\left.\begin{array}{l} 2y-1=0 \\ -2x+1=0 \end{array}\right\} \to \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

on bdy so "redundant"

Left $(x=0)$ $\quad f(0,y) = y^2 - y$

$\to 2y - 1 = 0 \to y = \frac{1}{2} \quad \left(0, \frac{1}{2}\right)$

Bottom $(y=0)$ $\quad f(x,0) = -x^2 + x$

$\to -2x + 1 = 0 \to x = \frac{1}{2} \quad \left(\frac{1}{2}, 0\right)$

Slant $(y=1-x)$ $\quad f(x,1-x) = (1-x)^2 - x^2 - (1-x) + x$

$= \cancel{1} - \cancel{2x} + \cancel{x^2} - \cancel{x^2} - \cancel{1} + \cancel{x} + \cancel{x}$

$= 0$

$\to$ function always 0 on $y = 1 - x$

since we include corners then don't need any additional points on this side

Corners $\quad (0,0), (0,1), (1,0)$

LIST

$f(0,0) = 0$

$f(0,1) = 0$

$f(1,0) = 0$

$f\left(0,\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$f\left(\frac{1}{2},0\right) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$

$$\boxed{\begin{array}{l} \text{absolute max} = \frac{1}{4} \\ \text{absolute min} = -\frac{1}{4} \end{array}}$$

score

1. Find the directional derivative of $h(x, y) = xy - e^{x-2y}$ at the point $(2, 1)$ in the same direction as the vector $\langle -4, 3\rangle$.

$$\nabla h(x,y) = \langle y - e^{x-2y}, x + 2e^{x-2y} \rangle$$

$$\nabla h(2,1) = \langle 1 - e^{0}, 2 + 2e^{0} \rangle = \langle 0, 4 \rangle$$

$$u = \frac{\langle -4, 3\rangle}{|\langle -4, 3\rangle|} = \frac{1}{5}\langle -4, 3\rangle$$

$$\nabla h(2,1) \cdot u = \langle 0, 4\rangle \cdot \frac{1}{5}\langle -4, 3\rangle = \boxed{\frac{12}{5}}$$

score

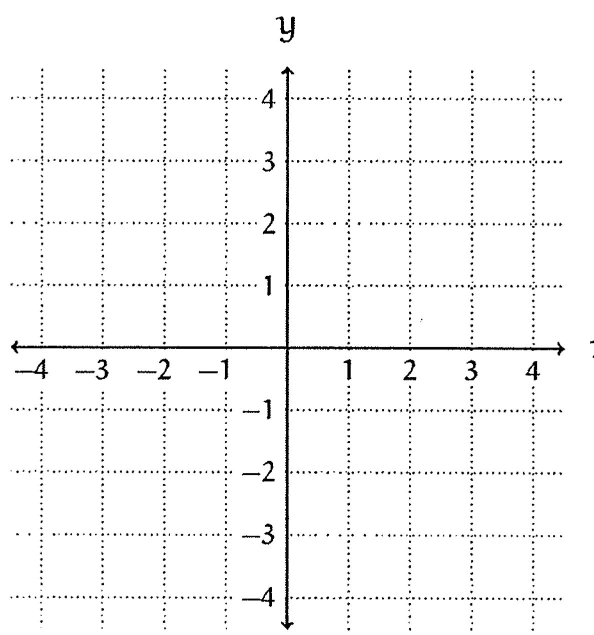
2. Consider the following function:

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{2x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

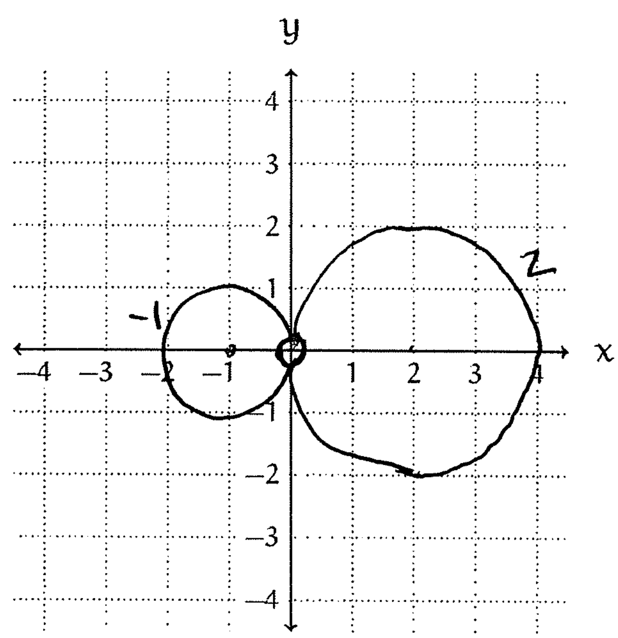
Note (0,0) has value 0 so not on either level curve

(a) Sketch and label the *level curves* for $f(x,y)$ for the values (or "heights") of -1 and 2.

For scratch work – *will not be graded*



Final answer – *will be graded*



$$\frac{x^2+y^2}{2x} = -1 \rightsquigarrow x^2+y^2 = -2x \rightsquigarrow x^2+2x+1+y^2 = 1 \rightarrow \underbrace{(x+1)^2+y^2 = 1}_{\text{circle, } (-1,0) \text{ radius } 1}$$

$$\frac{x^2+y^2}{2x} = 2 \rightsquigarrow x^2+y^2 = 4x \rightsquigarrow x^2-4x+4+y^2 = 4 \rightarrow \underbrace{(x-2)^2+y^2 = 2^2}_{\text{circle, } (2,0) \text{ radius } 2}$$

(b) Determine whether the $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and if so give its value, if not explain (briefly!) why the limit does not exist.

Limit does not exist since we can get different values by traveling on level curves from (a)

score

4. Let $f(x,y) = \frac{1}{x} + xy + \frac{1}{y}$ where $x \neq 0$ and $y \neq 0$.

(a) Find the critical point(s) of $f(x,y)$.

$\nabla f = 0$

$f_x = -\frac{1}{x^2} + y = 0 \rightsquigarrow y = \frac{1}{x^2}$

$f_y = x - \frac{1}{y^2} = 0 \rightsquigarrow x = \frac{1}{y^2} = \frac{1}{\left(\frac{1}{x^2}\right)^2} = x^4$

$\rightsquigarrow x^4 - x = 0$

$\rightsquigarrow x(x^3-1) = 0$

$\boxed{(1,1)}$

$x = 0$ impossible (since $x \neq 0$)

$x^3 = 1$ or $x = 1$ then $y = \frac{1}{1^2} = 1$ $(1,1)$

(b) Classify each critical point as a local maximum, local minimum, or a saddle point.

$f_{xx} = \frac{2}{x^3}$

$f_{yy} = \frac{2}{y^3}$

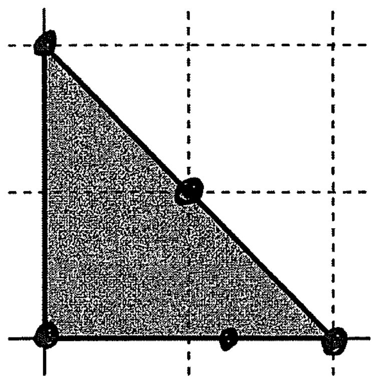
$f_{xy} = 1$

$D = f_{xx}f_{yy} - (f_{xy})^2 = \frac{4}{x^3y^3} - 1$

$\left.\begin{array}{l} D(1,1) = 4 - 1 > 0 \\ f_{xx}(1,1) = 2 > 0 \end{array}\right\}$ $\boxed{\text{LOCAL MIN at } (1,1)}$

score

5. Find the absolute maximum and minimum of $f(x,y) = 2x^2 + xy - 5x + \frac{1}{2}y^2 - 2y + 4$ on or inside the triangle with vertices $(0,0)$, $(2,0)$, and $(0,2)$.



Interior

$f_x = 4x + y - 5 = 0$

$-(f_y = x + y - 2 = 0)$ $\times(-1)$

$3x + 0 - 3 = 0 \rightsquigarrow x = 1$
$y = 2 - x \longrightarrow y = 1$ $\Big\} (1,1)$

LIST

$f(1,1) = 1/2$

$f(0,2) = 2$

$f(\frac{5}{4},0) = \frac{7}{8}$

$f(0,0) = 4$

$f(2,0) = 2$

$\boxed{\begin{array}{l} MAX = 4 \\ MIN = \frac{1}{2} \end{array}}$

score

Left $(0,y)$

$g(y) = f(0,y) = \frac{1}{2}y^2 - 2y + 4$

$g'(y) = y - 2 \rightsquigarrow y = 2 \rightsquigarrow (0,2)$

Bottom $(x,0)$

$h(x) = f(x,0) = 2x^2 - 5x + 4$

$h'(x) = 4x - 5 = 0 \rightsquigarrow x = \frac{5}{4} \rightsquigarrow (\frac{5}{4}, 0)$

Slant $(x, 2-x)$

$k(x) = f(x,2-x) = 2x^2 + x(2-x) - 5x + \frac{1}{2}(2-x)^2 - 2(2-x) + 4$

$= 2x^2 + \cancel{2x} - x^2 - 5x + 2 - \cancel{2x} + \frac{1}{2}x^2 \cancel{-4} + 2x \cancel{+4}$

$= \frac{3}{2}x^2 - 3x + 2$

$k'(x) = 3x - 3 = 0 \rightsquigarrow x = 1 \rightsquigarrow (1,1)$

Corners

$(0,0), (0,2), (2,0)$

1. Let $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

(a) Find all the critical points of $f(x,y)$.

$F_x(x,y) = 3x^2 + 6x = 3x(x+2) = 0 \rightsquigarrow x = 0$ or $x = -2$

$F_y(x,y) = 3y^2 - 6y = 3y(y-2) = 0 \rightsquigarrow y = 0$ or $y = 2$

pair up in all possible ways

$\boxed{\begin{array}{l}(0,0)\\(0,2)\\(-2,0)\\(-2,2)\end{array}}$

(b) Classify each of the critical points found in part (a) as local maxima, local minima, or saddle points.

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (6x+6)(6y-6) - 0$

$D(0,0) = -36 \rightsquigarrow \boxed{(0,0) \text{ Saddle}}$

$D(0,2) = 36$ and $f_{xx}(0,2) = 6 > 0 \rightsquigarrow \boxed{(0,2) \text{ local min}}$

$D(-2,0) = 36$ and $f_{xx}(-2,0) = -6 < 0 \rightsquigarrow \boxed{(-2,0) \text{ local max}}$

$D(-2,2) = -36 \rightsquigarrow \boxed{(-2,2) \text{ saddle}}$

score

2. Find the following limits, or show they do not exist:

(a) As $(x, y) \to (0, 0)$ of $\dfrac{\sec(x+y)\ln(2-x)}{\sqrt{9-\pi e^{x^2+y^2}}}$.

$$\lim_{(x,y)\to(0,0)} \frac{\sec(x+y)\ln(2-x)}{\sqrt{9-\pi e^{x^2+y^2}}} = \frac{\overset{1}{\sec(0)}\ln(2)}{\sqrt{9-\pi e^0}} = \boxed{\frac{\ln 2}{\sqrt{9-\pi}}}$$

(b) As $(x, y) \to (0, 0)$ of $x^2 \sin^2(\ln(|x| + |y|))$.

blows up at $(0,0)$ so this will cause frequent oscillation (frequent but bounded)

$$0 \le \sin^2(\ln(|x|+|y|)) \le 1$$

$$0 \le x^2\sin^2(\ln(|x|+|y|)) \le x^2$$

$\lim_{(x,y)\to(0,0)} 0 = 0$ $\qquad$ $\lim_{(x,y)\to(0,0)} x^2 = 0$

By squeeze we conclude that

$$\lim_{(x,y)\to(0,0)} x^2\sin^2(\ln(|x|+|y|)) = \boxed{0}$$

score

6. Kelly is in the business of producing different types of widgets and gadgets. For every g gadgets and w widgets produced, Kelly's profit is given by $P(g, w) = 30g^{1/3}w^{2/3}$. Each gadget costs \$2 to produce and each widget costs \$1 to produce. If Kelly only has \$300 to spend on production, how many gadgets and widgets should be produced to maximize profit?

Constraint: $2g + w = 300$

Optimize: $30g^{1/3}w^{2/3}$

$$\frac{\partial P}{\partial g} = 10g^{-2/3}w^{2/3} = \lambda \frac{\partial}{\partial g}(2g+w) = 2\lambda$$

$$\frac{\partial P}{\partial w} = 20g^{1/3}w^{-1/3} = \lambda \frac{\partial}{\partial w}(2g+w) = \lambda$$

$$\rightsquigarrow \quad 10g^{-2/3}w^{2/3} = 2\lambda = 40g^{1/3}w^{-1/3}$$

$$\rightsquigarrow \quad \frac{w^{2/3}}{g^{2/3}} = 4\frac{g^{1/3}}{w^{1/3}}$$

$$\rightsquigarrow \quad w = 4g$$

Putting this into the constraint we have

$$2g + 4g = 300$$

$$\rightsquigarrow 6g = 300$$

$$\rightsquigarrow g = 50 \quad \text{and} \quad w = 4g = 200$$

$$\boxed{\begin{array}{c}\text{50 gadgets} \\ \text{200 widgets}\end{array}}$$

score

2. Given that $f(x,y) = g\big(u(x,y), v(x,y)\big)$ and the following information, determine the tangent plane to $f(x,y)$ at the point $(3,1)$.

(a,b)	$g(a,b)$	$g_u(a,b)$	$g_v(a,b)$	$u(a,b)$	$u_x(a,b)$	$u_y(a,b)$	$v(a,b)$	$v_x(a,b)$	$v_y(a,b)$
(1,3)	−2	3	−1	2	5	4	−3	5	6
(1,4)	1	−4	7	2	6	4	5	9	1
(3,1)	−5	2	−8	1	7	2	4	6	3
(3,4)	3	1	2	−5	6	2	9	−5	7

Since $u(3,1) = 1$ and $v(3,1) = 4$, we have $f(3,1) = g\big(u(3,1), v(3,1)\big) = g(1,4) = 1$. To find the partial derivatives we use the chain rule, namely

$$f_x(x,y) = g_u\big(u(x,y), v(x,y)\big)u_x(x,y) + g_v\big(u(x,y), v(x,y)\big)v_x(x,y), \text{ and}$$
$$f_y(x,y) = g_u\big(u(x,y), v(x,y)\big)u_y(x,y) + g_v\big(u(x,y), v(x,y)\big)v_y(x,y).$$

Evaluating at $(3,1)$ we get

$$f_x(3,1) = g_u(1,4)u_x(3,1) + g_v(1,4)v_x(3,1) = (-4)(7) + (7)(6) = -28 + 42 = 14, \text{ and}$$
$$f_y(3,1) = g_u(1,4)u_y(3,1) + g_v(1,4)v_y(3,1) = (-4)(2) + (7)(3) = -8 + 21 = 13.$$

Putting this together we get the tangent plane

$$z = f(3,1) + f_x(3,1)(x-3) + f_y(3,1)(y-1) \text{ giving } z = 1 + 14(x-3) + 13(y-1).$$

3. Find and classify all of the critical points for $f(x,y) = x^4 - 2x^2y^2 + y^4 + \frac{8}{3}y^3 - 16y^2$.

We have $f_x(x,y) = 4x^3 - 4xy^2 = 4x(x^2 - y^2) = 4x(x-y)(x+y) = 0$ and $f_y(x,y) = -4x^2y + 4y^3 + 8y^2 - 32y = 0$. From f_x we get that either $x = 0$, $x = y$, or $x = -y$. From $x = 0$ into f_y we get $0 = 4y^3 + 8y^2 - 32y = 4y(y^2 + 2y - 8) = 2y(y+4)(y-2)$ so $y = 0$, -4, or 2 giving the points $(0,0)$, $(0,-4)$ and $(0,2)$. From $x = y$ into f_y we get $0 = 8y^2 - 32y = 8y(y-4)$ so $y = 0$ or 4 giving the points $(0,0)$ and $(4,4)$. From $x = -y$ into f_y we get $0 = 8y^2 - 32y = 8y(y-4)$ so $y = 0$ or 4 giving the points $(0,0)$ and $(-4,4)$. So we have five critical points: $(0,0)$, $(0,-4)$, $(0,2)$, $(4,4)$, and $(-4,4)$.

We have $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (12x^2 - 4y^2)(-4x^2 + 12y^2 + 16y - 32) - (-8xy)^2$.

$D(0,0) = (0)(-32) - (0)^2 = 0$ so we do not have enough information to classify $(0,0)$.
$D(0,-4) = (-64)(96) - (0)^2 < 0$ so $(0,-4)$ is a saddle.
$D(0,2) = (-16)(48) - (0)^2 < 0$ so $(0,2)$ is a saddle.
$D(4,4) = (128)(160) - (128)^2 > 0$ and $f_{xx}(4,4) = 128 > 0$ so $(4,4)$ is a minimum.
$D(-4,4) = (128)(160) - (128)^2 > 0$ and $f_{xx}(-4,4) = 128 > 0$ so $(-4,4)$ is a minimum.

(Note that it is not hard to show that $(0,0)$ is a saddle by looking at the function along the x-axis and then along the y-axis. But this is not needed.)

4. Given that $f(s,t) = 2e^{s-3t} + s^2 - 4t$ use the linear approximation of $f(s,t)$ at the point $(3,1)$ to estimate $f(3.1, 1.1)$.

The linear approximation is another name for the tangent plane. We have $f_s(s,t) = 2e^{s-3t} + 2s$ and $f_t(s,t) = -6e^{s-3t} - 4$ so that $f_s(3,1) = 8$ and $f_t(3,1) = -10$. So the tangent plane is

$$z = f(3,1) + f_s(3,1)(s-3) + f_t(3,1)(t-1) \text{ giving } z = 7 + 8(s-3) - 10(t-1).$$

So we can conclude $f(3.1, 1.1) \approx 7 + 8(0.1) - 10(0.1) \approx 6.8$.

5. Find the rate of change of $f(x,y,z) = x^2z + \ln(xy^2z^3)$ at $(1,1,1)$ in the direction $\langle -4,7,4\rangle$.

This is a directional derivative at the point $(1,1,1)$. To find the unit vector we take our current vector and scale by dividing by $|\langle -4,7,4\rangle| = \sqrt{16+49+16} = \sqrt{81} = 9$ so that $\mathbf{u} = \frac{1}{9}\langle -4,7,4\rangle$. Finally we compute our gradient, but first note that by properties of the logarithm we can write our function as $f(x,y,z) = x^2z + \ln(x) + 2\ln(y) + 3\ln(z)$. So we have $\nabla f(x,y,z) = \langle 2xz + \frac{1}{x}, \frac{2}{y}, x^2 + \frac{3}{z}\rangle$ and $\nabla f(1,1,1) = \langle 3,2,4\rangle$. So we can conclude that

$$D_{\mathbf{u}}f(p) = \nabla f(1,1,1)\cdot \mathbf{u} = \langle 3,2,4\rangle \cdot \frac{1}{9}\langle -4,7,4\rangle = \frac{1}{9}(-12+14+16) = 2.$$

---

6. Find the maximum and minimum value of $f(x,y) = x^2 + 4y^2$ for the closed and bounded region of the points satisfying $5x^2 + 12xy + 20y^2 \leqslant 64$.

This is optimization on a closed and bounded set. First we look for interior points which are $\nabla f = \langle 2x, 8y\rangle = \langle 0,0\rangle$ which gives us the point $(0,0)$ (which is clearly in our set of points).

Now we look on the boundary which are the points $5x^2 + 12xy + 20y^2 = 64$. For this we now transition to the method of Lagrange multipliers where $g(x,y) = 5x^2 + 12xy + 20y^2$ and $\nabla g(x,y) = \langle 10x + 12y, 12x + 40y\rangle$. We have $\nabla f = \lambda \nabla g$ or $f_x = \lambda g_x$ and $f_y = \lambda g_y$. In particular we can conclude

$$\frac{2x}{10x+12y} = \frac{f_x}{g_x} = \lambda = \frac{f_y}{g_y} = \frac{8y}{12x+40y}.$$

Cross-multiplying we have $24x^2 + 80xy = 80xy + 96y^2$ or $x^2 = 4y^2$ or $x = 2y$ and $x = -2y$. Putting $x = 2y$ back into our constraint gives $20y^2 + 24y^2 + 20y^2 = 64$ or $y^2 = 1$ or $y = \pm 1$ giving $(2,1)$ and $(-2,-1)$. Putting $x = -2y$ back into our constraint gives $20y^2 - 24y^2 + 20y^2 = 64$ or $y^2 = 4$ or $y = \pm 2$ giving $(-4,2)$ and $(4,-2)$.

We now have five points and so we evaluate each.

$$\begin{aligned} f(0,0) &= 0 \\ f(2,1) &= 8 \\ f(-2,-1) &= 8 \\ f(-4,2) &= 32 \\ f(4,-2) &= 32 \end{aligned}$$

So the minimum value is 0 and the maximum value is 32.

5. Use the method of **Lagrange multipliers** to find the maximum of $2x - y$ given that $3x^2 - 4xy + 2y^2 = 6$. Also give the point (x, y) where the maximum is achieved. (The curve $3x^2 - 4xy + 2y^2 = 6$ is a closed and bounded set so the maximum must be achieved.)

We have $f(x, y) = 2x - y$ with $\nabla f = \langle 2, -1 \rangle$ and $g(x, y) = 3x^2 - 4xy + 2y^2$ with $\nabla g = \langle 6x - 4y, -4x + 4y \rangle$. We have $\nabla f = \lambda \nabla g$ or $f_x = \lambda g_x$ and $f_y = \lambda g_y$. In particular we can conclude

$$\frac{2}{6x - 4y} = \frac{f_x}{g_x} = \lambda = \frac{f_y}{g_y} = \frac{-1}{-4x + 4y}.$$

From this we have that $-8x + 8y = -6x + 4y$ or $4y = 2x$ or $x = 2y$. Putting this into the constraint we have $6 = 12y^2 - 8y^2 + 2y^2 = 6y^2$ so that $y = 1$ or $y = -1$ giving the possible points $(2, 1)$ and $(-2, -1)$. Since $f(2, 1) = 3$ and $f(-2, -1) = -3$ we can conclude that the maximum is 3 and is achieved at the point $(2, 1)$.

---

5. Use the method of **Lagrange multipliers** to find the minimum of $4x^2 - 3xy + 2y^2$ given that $2x + 5y = 12$. Also give the point (x, y) where the minimum is achieved. (You do not have to prove that it is a minimum.)

We have $f(x, y) = 4x^2 - 3xy + 2y^2$ with $\nabla f = \langle 8x - 3y, -3x + 4y \rangle$ and $g(x, y) = 2x + 5y$ with $\nabla g = \langle 2, 5 \rangle$. We have $\nabla f = \lambda \nabla g$ or $f_x = \lambda g_x$ and $f_y = \lambda g_y$. In particular we can conclude

$$\frac{8x - 3y}{2} = \frac{f_x}{g_x} = \lambda = \frac{f_y}{g_y} = \frac{-3x + 4y}{5}.$$

From this we have that $40x - 15y = -6x + 8y$ or $46x = 23y$ or $y = 2x$. Putting this into the constraint we have $2x + 10x = 12$ so that $x = 1$ giving the point $(1, 2)$. Since $f(1, 2) = 6$ we can conclude that the minimum is 6 and is achieved at the point $(1, 2)$.