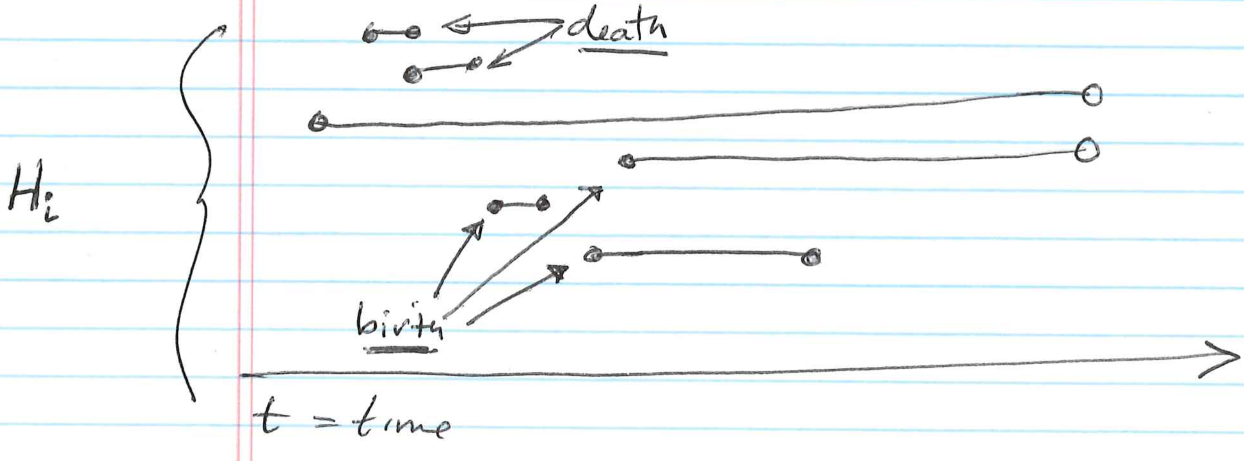
Lecture 2

TDA

Preamble: For this lecture and next lecture, we will be moving towards the following goal.

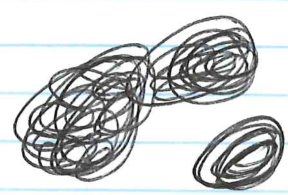
"Fundamental Thm of PH" (FTPH)

PH is completely encoded by a barcode, consisting of long bars and short bars.

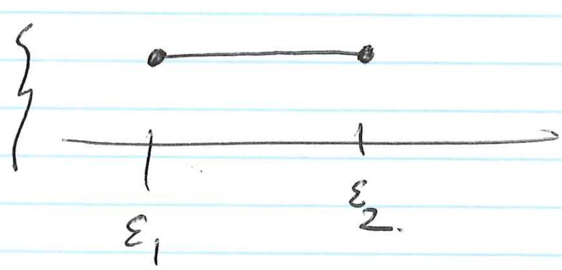


Intuition in our "idea" from last lecture, we saw at some time (birth) ε_1, our diagram was

$\{H_1 \neq 0\} \longleftarrow S^1 \longrightarrow$ 

and at a later ε_2, we had 

So the H_1 barcode would be



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FTPH is a consequence of the following

[MPID] **Theorem** (Artin, 6.13 or Fraleigh, 11.12)

A finitely generated module M over a Euclidean Domain R (more generally, over a PID) has a decomposition

$$M \simeq R^n \oplus R/_{d_i}, \quad d_i \mid d_{i+1} \qquad \{d_i \neq 0, \text{unit}\}$$

Basic idea: Long bars $\Leftrightarrow$ free summands $\Leftrightarrow R^n$
Short bars $\Leftrightarrow$ torsion $\Leftrightarrow \oplus R/d_i$

Amazing and Beautiful Fact MPID has very different avatars

1. 11.12 (Fraleigh). A finitely generated Abelian group is $\simeq \mathbb{Z}^n \oplus \bigoplus_i \mathbb{Z}/_{(p_i)^{\alpha_i}}$ p_i prime

2. 12.7.9 (Artin). A linear operator on a finite dim vector space can be written (Δ basis) as a block diagonal matrix, with blocks

$$\underbrace{\begin{bmatrix} 0 & & & -a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \vdots \\ & & 1 & -a_n \end{bmatrix}}_{\text{square.}}$$

A matrix of this type is said to be in <u>rational canonical form</u>: it is the ~~closest~~ closest we can get to diagonalization in general
(not every matrix can be diagonalized)

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Two concepts appeared in the statement of the theorem

DEF If M_1, M_2 are R-modules, then

$M_1 \oplus M_2 = \{ (m_1, m_2) \mid m_i \in M_i \}$ is an R-module,

with $(m_1, m_2) + (n_1, n_2) = (m_1 + n_1, m_2 + n_2)$; $r(m_1, m_2) = (rm_1, rm_2)$

DEF • An integral domain R is a principal ideal (PID) domain if every ideal $\cancel{I} \subseteq R$ is of the form $I = \langle m \rangle$ $\{$ is principal $=$ 1 generator $\}$

• An integral domain R is a Euclidean domain (ED) if it possesses a Euclidean norm (EN)

SUBDEF: A EN on a domain R is a function

$$R \setminus 0 \xrightarrow{v} \mathbb{Z}_{\geq 0} \quad \text{such that}$$

- $\forall\, a, b \in R,\ b \neq 0\ \exists\, q, r \in R$ s.t. $a = bq + r$, with $r = 0$ or $v(r) < v(b)$
- $\forall\, a \neq 0 \neq b \in R \quad v(a) \leq v(ab)$

Examples • $R = \mathbb{Z}$, $v(n) = |n|$

• $R = k[x]$, $v(f) = \deg(f)$

Core idea: $\{$ A Euclidean domain gives us a division algorithm $\}$

THM ED is a PID.

PF: Let I be an ideal. $I = \langle 0 \rangle$, done, if not, choose $f \in I$ with $v(f)$ minimal. Any $g \in I$ satisfies $g = qf + r$, $r = 0$ or $v(r) < v(f)$. But $r = g - fq \in I$, so can't have $v(r) < v(f) \Rightarrow r = 0$ $\blacksquare$

18 seconds

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REST OF TODAYS LECTURE — RATIONAL CANONICAL FORM

Turbo Lin Alg Review (Next 20 minutes)

Let V be a vector space of finite dim (vsfd) over $\underset{\text{field}}{k}$

DEF
- $\{v_1 \ldots v_k\}$ spans V if any $v \in V$ can be written $v = \sum_i^k a_i v_i$
- " is linearly indep (LID) if $\sum_i^k a_i v_i = 0 \Rightarrow$ all $a_i = 0$
- " is a basis if it spans and is LID.

Exercise: Cardinality of a basis (vsfd) is well defined.

DEF A map $V \xrightarrow{T} V$ is a linear transformation (LT) if $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2)$, $v_i \in V$, $\alpha \in k$.

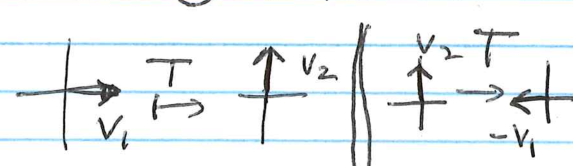
Exercise A LT $V \to V \cong k^n$ may be written as an $n \times n$ matrix, via the procedure
$\beta = \{v_1 \ldots v_n\}$ basis for V

$$A_\beta = \left[T(v_1)_\beta \,\middle|\, \cdots \,\middle|\, T(v_n)_\beta \right]$$

where $T(v_i)_\beta$ means apply $T(v_i)$, write result in terms of β.

Example: $T: \mathbb{R}^2 \to \mathbb{R}^2$ via $90^\circ \circlearrowleft$ rotation.

$$B = \{ \underset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}0\\1\end{pmatrix}} \} \begin{matrix} x \\ y \end{matrix}$$



So $T_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Check that using $B' = \{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \}$ gives

$$T_{B'} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

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WHO CARES? Suppose T represents a Markov process, and we want to do the process n times

$$(T)^n = A^n = \boxed{\text{very expensive \$\$}} \longleftarrow$$

If we choose (or are lucky) a basis where A is diagonal, it is trivial: $\begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_m \end{pmatrix}^n = \begin{pmatrix} \alpha_1^n & & 0 \\ & \ddots & \\ 0 & & \alpha_m^n \end{pmatrix}$ ☺

DEF v is an eigenvector for T if $(T-\lambda I)v = 0$, λ is the eigenvalue, and $Tv = \lambda v$, so if v is part of a basis β, $(T(v))_\beta = \begin{pmatrix} \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ (say v = first basis vect)

Hence, if T has a basis of eigenvectors, T is diagonalizable ☺

Exercise Find an invertible 2×2 matrix which can't be diagonalized

Wake up: Lin alg review is over!

Rational Canonical Form: (Artin §12.7, Lang "Algebra" XIV.2)

DEF: An invariant block is an $n\times n$ square matrix

$$B_n = \begin{bmatrix} 0 & \cdots & \cdots & 0 & a_1 \\ 1 & 0 & & \vdots & \vdots \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & & \ddots & & \\ 0 & \cdots & 0 & 1 & a_n \end{bmatrix}$$

= 1's (immediately) below diagonal, 0's elsewhere except last col, $a_i \in K$.

Theorem Any $n\times n$ matrix is similar (= after a basis) to a matrix with $\sum_{j=1}^{k} i_j = n$ and B_{i_j} invariant $\Longrightarrow$

$$\begin{bmatrix} B_{i_1} & & & \\ & B_{i_2} & & \\ & & \ddots & \\ & & & B_{i_k} \end{bmatrix}$$

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Pf: Let $v \in V$, T a linear transfm $V \xrightarrow{T} V$

let j be the smallest integer such that

$T^j(v) \in \text{span}\{v, T(v), T^2(v) \dots T^{j-1}(v)\} = \beta$.

Put $V_{j-1} = \text{span}(\beta)$, β is clearly a basis for V_{j-1}

V_{j-1} is a subspace of V, and wrt β,

$$\left[T|_{V_{j-1}}\right]_\beta = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & a_0 \\ 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & 0 & \vdots \\ \vdots & & 1 & a_{j-2} \\ 0 & \cdots & 0 & a_{j-1} \end{bmatrix}}_{\text{an invariant block.}} \text{, where } T^j(v) = \sum_{i=0}^{j-1} a_i T^i(v)$$

First col. $\leftarrow$ $j \times j$ matrix

$v_1 = v \to T(v) = v_2$.

Now just iterate the process for V/V_{j-1} ∎

Connecting Linear transforms to $K[t]$ modules:

Thm (Artin, p476): A linear operator T on a vsfd V/k and a $k[t]$ module (f'gend) are equiv.

Sketch Pf: • Let $T: V \to V$ a LT, and $f(t) = \sum_{i=0}^{m} a_i t^i$

we make V into a $K[t]$-module via

$$f(t) \cdot v = \sum_{i=0}^{m} a_i T^i(v).$$

Check this operation satisfies the module requirements.

• If V is a $k[t]$ module, then K acts on V, so V is a K-vector space, and $\cdot t$ maps $V \to V$. The module properties mean $\cdot t$ is a linear op. so define $T = \cdot t$

∎

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Thus, we have shown

(1) T a linear operator on V vsfd $/k$

$$\Updownarrow$$

V a finitely generated $k[t]$ module

remember, this is a Euclidean Domain!

(2) Any lin. op T on a vsfd $/k$ has a decomposition into a rational canonical form

$$\begin{bmatrix} B_{i_1} & 0 & 0 & 0 \\ 0 & B_{i_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & B_{i_k} \end{bmatrix}$$

With B_{i_j} square invariant blocks

(3) Concrete example: suppose $V = k[t]/\langle f(t)\rangle$, where $f(t) = t^n + a_{n-1}t^{n-1} + \dots + a_0$. Clearly V is a vsfd over k, with basis

$$1, t, \dots, t^{n-1}$$

$$\downarrow T \qquad \downarrow$$

$$t \quad t^2 \quad \dots \quad t^n = -\sum_{j=0}^{n-1} a_i t^i$$

So matrix is
$$\begin{bmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & \cdots & \vdots & \vdots \\ 0 & & \vdots & \vdots \\ \vdots & & 0 & \vdots \\ 0 & & 1 & -a_{n-1} \end{bmatrix}$$
← invariant!

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Jordan Canonical Form

Suppose $f(t) = (t-\alpha)^n$, $W = {}^{k[t]}/_{\langle f(t)\rangle}$

We use a slightly different basis

$w_0 :=$ residue class of 1 in W

$w_i := (t-\alpha)^i w_0$

Then $(t-\alpha) w_0 = w_1$

$\vdots$

$(t-\alpha) w_{n-2} = w_{n-1}$ and $(t-\alpha) w_{n-1} = 0$

Hence $(T-\alpha) w_i = w_{i+1} \Rightarrow \left. T w_i = \alpha w_i + w_{i+1} \right\} i = 0 \ldots n-2$

$T w_{n-1} = \alpha w_{n-1}$

So the matrix for T is $\begin{bmatrix} \alpha & 0 & \cdots & \cdots & 0 \\ 1 & \alpha & & & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \vdots & 0 & 1 & \alpha \end{bmatrix}$ $\Leftarrow$ Jordan Block

Notice: If k is alg closed (say, if $k = \mathbb{C}$)

then any $f(t) = \prod_{i=1}^{n} (t-\alpha_i)^{m_i}$

So we get a representation of T w/ Jordan block

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BONUS: Googles PageRank Algorithm

① Represent the web as a weighted, directed graph

vertices = website

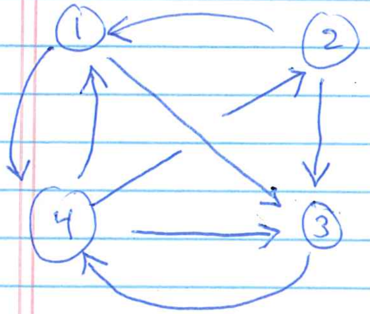
edge $i \to j$ if site i has link to site j

weight: if site i has k links = arrows pointing "out"

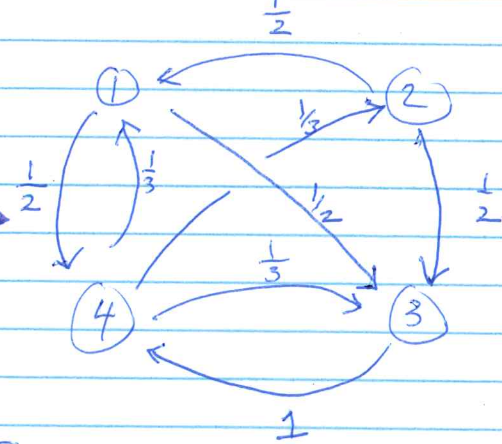
then each arrow out of i has weight $\frac{1}{k}$

intuition: If you're at site i, equally likely to choose one of k arrows out.

Example Site 1 links to 3, 4
2 " 1, 3
Lecture 2 3 " 4
4 " 1, 2, 3



weighting, we get $\Longrightarrow$



(2) From the graph, build a transition matrix:

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 1 & \frac{1}{3} \end{bmatrix} =: T$$

T is $n \times n$, $n = \#$ websites
If v_m is $n \times 1$, $\Sigma v_i = 1$,
then $T(v_m) = v_{m+1}$ = probabilities at time $m+1$

(3) But, there is a probability you don't follow a link, but type in a new URL. Assume jumping to <u>any</u> site is equally likely.

[Aside in practice, $p \approx .15$]

<u>DEF</u> $G = (1-p)T + p\left(\frac{1}{n}\hat{\mathbb{1}}\right)$, $\hat{\mathbb{1}}$ is $n \times n$ matrix with all 1's
$p \in [0, 1]$

<u>Rmk</u>: G is positive + column stochastic (all cols sum to 1)

<u>THM</u> (Perron-Frobenius) A positive, column stochastic matrix has 1 as an eval; corr. evect v^* is all +.

<u>THM</u> If v^* is as above, and $v(0) = \frac{1}{n}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, then

$$\lim_{n \to \infty} G^n(v(0)) = v^* \leftarrow \text{the page rank!}$$

<u>ALG</u>: To find PageRank, solve $Gv = 1v$, scale v so entries sum to 1.

Problem: n is in the billions!

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Exercises

1. Prove if V vsfd $/ k$ then the cardinality of a basis for V is well defined. DEF $|\text{Basis}| = \dim V$

2. Find the change of basis matrix $\Delta_{B'B} : B \to B'$ for example on P3. (notes)

   Show $\Delta_{B'B} \cdot \Delta_{BB'} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3. Find the rational canonical form for $V = K[t] / t^3 - 4t^2 + 5t - 2 = f(t)$. Hint: factor $f(t)$

4. Diagonalize the matrix T on P8 (notes)