Lecture 2

TDA

Preamble: For this lecture and next lecture, we will be moving towards the following goal:

"Fundamental Thm of PH" (FP Thm)

PH is completely encoded by a barcode, consisting of long bars and short bars:

\[ \text{birth} \rightarrow \text{death} \]

\( t = \text{time} \)

Intuition in our "idea" from last lecture, we saw at some time (birth) \( \varepsilon_1 \), our diagram was

\[ S^1 \not\cong \mathbb{R}^2 \]

and at a later \( \varepsilon_2 \), we had so the \( H_1 \) barcode would be

\[ \varepsilon_1 \quad \varepsilon_2 \]

\[ \varepsilon_1 \rightarrow \varepsilon_2 \]
FTP is a consequence of the following

\[ \text{Theorem (Artin, 6.13 or Fraleigh, 11.12)} \]

A finitely generated module \( M \) over a Euclidean Domain \( R \) (more generally, over a PID) has a decomposition

\[ M \cong R^n \oplus \bigoplus_{i} \frac{R}{d_i}, \quad d_i \neq 0, \text{unit} \]

Basic idea: Long bars \( \equiv \) free summands \( \cong R^n \)
Short bars \( \equiv \) torsion \( \cong \bigoplus \frac{R}{d_i} \)

Amazing and Beautiful Fact: MPID has very different avatars

1. 11.12 (Fraleigh) A finitely generated Abelian group \( G \cong \mathbb{Z}^n \oplus \bigoplus_{i} \frac{\mathbb{Z}}{(p_i)} \) where \( p_i \) prime

2. 12.7.9 (Artin) A linear operator on a finite dimensional vector space can be written (a basis) as a block diagonal matrix, with blocks

\[
\begin{bmatrix}
0 & \lambda \\
\omega & 0 \\
\end{bmatrix}
\]

A matrix of this type is said to be in rational canonical form: it is the closest we can get to diagonalized in general (not every matrix can be diagonalized).
Two concepts appeared in the statement of the theorem.

**DEF** If $M_1, M_2$ are $R$-modules, then

$$M_1 \oplus M_2 = \{ (m_1, m_2) \mid m_1 \in M_1 \}$$

is an $R$-module, with

$$\begin{align*}
(m_1 + n_1, m_2 + n_2) &= (m_1 + n_1, m_2 + n_2) \\
r(m_1, m_2) &= (rm_1, rm_2)
\end{align*}$$

**DEF.** An integral domain $R$ is a [principal ideal domain (PID)](https://en.wikipedia.org/wiki/Principal_ideal_domain) if every ideal $I \subseteq R$ is of the form $I = \langle m \rangle$ is principal = 1 generator.

- An integral domain $R$ is a [Euclidean domain (ED)](https://en.wikipedia.org/wiki/Euclidean_domain) if it possesses a Euclidean norm (EN).

**SUBDEF:** A Euclidean norm on a domain $R$ is a function $\nu : R \setminus \{0\} \to \mathbb{Z}_{>0}$ such that

- $\forall a, b \in R, b \neq 0 \exists q, r \in R$ s.t.
  $a = bq + r$, with $r = 0$ or $\nu(r) < \nu(b)$
- $\forall a \neq 0 \in R \forall a \leq \nu(ab)$

**Examples**

- $R = \mathbb{Z}$, $\nu(n) = |n|$
- $R = \mathbb{R}[x]$, $\nu(f) = \deg(f)$

**Core idea:** A Euclidean domain gives us a division algorithm.

**THM** ED is a PID.

**PF:** Let $I$ be an ideal. $I = \langle 0 \rangle$ done, if not, choose $f \in I$ with $\nu(f)$ minimal. Any $g \in I$ satisfies $g = qf + r$, $r = 0$ or $\nu(r) < \nu(f)$. But $r = q - f \in I$, so can't have $\nu(r) < \nu(f) \Rightarrow r = 0$.
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REST OF TODAY'S LECTURE - RATIONAL CANONICAL FORM

Turbo Lin Alg Review (Next 20 minutes)

Let $V$ be a vector space of finite dim (vsfd) over $\mathbb{K}$

**DEF** $\{v_1, \ldots, v_k\}$ spans $V$ if any $v \in V$ can be written $v = \sum_i a_i v_i$

is linearly indep (LID) if $\sum_i a_i v_i = 0 \Rightarrow a_i = 0$

is a basis if it spans and is LID.

**Exercise:** Cardinality of a basis (vsfd) is well defined.

**DEF** A map $V \to V$ is a linear transformation (LT) if $T(\alpha v_1 + v_2) = \alpha T(v_1) + T(v_2)$, $v_i \in V, \alpha \in \mathbb{K}$

**Exercise** A LT $V \to V \cong \mathbb{K}^n$ may be written as an $n \times n$ matrix, via the procedure

$\beta = \{v_1, \ldots, v_n\}$ basis for $V$

$A_\beta = \begin{bmatrix} T(v_1) \\ \vdots \\ T(v_n) \end{bmatrix}$

where $T(v_i)_\beta$ means apply $T(v_i)$, write result in terms of $\beta$.

**Example:** $T: \mathbb{R}^2 \to \mathbb{R}^2$ via $90^\circ$ rotation.

$\beta = \{ (1, 0), (0, 1) \}$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} v_2 \\ -v_1 \end{bmatrix}$

So $T_\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Check that using $\beta' = \{ (0, 1), (1, 0) \}$ gives

$T_{\beta'} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
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WHO CARES? Suppose $T$ represents a Markov process, and we want to do the process $n$ times.

$$(T^n) = A^n = \begin{bmatrix} \text{very} & \# \\ \text{expensive} & \end{bmatrix} \quad \text{Markov process, and we want}$$

to do the process $n$ times.

If we choose (or are lucky) a basis where $A$ is diagonal, it is trivial:

$$\begin{bmatrix} x_i & 0 \\ 0 & x_m \end{bmatrix}^n = \begin{bmatrix} x_i^n & 0 \\ 0 & x_m^n \end{bmatrix} \quad \text{(1)}$$

DEF: $v$ is an eigenvector for $T$ if $(T - \lambda I)v = 0$.

$\lambda$ is the eigenvalue, and $Tv = \lambda v$, so if $v$ is

part of a basis $\mathcal{B}$, $T(v) = (\lambda \cdot v)$ (say $v$ = first

basis vector).

Hence, if $T$ has a basis of eigenvectors, $T$ is diagonalizable. \(\text{(2)}\)

Exercise: Find an invertible $2 \times 2$ matrix which diagonalized $T$.

Wake up: Lin alg review is over!

Rational Canonical Form: (Artin §12.7, Lang Algebra XIV.2)

DEF: An invariant block is an $n \times n$ square matrix

$$B_n = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Theorem: Any $n \times n$ matrix is similar (i.e., after a basis)
to a matrix

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \quad \text{with} \quad \sum_{j=1}^{n} i_{j} = n$$

and $B_{i,j}$ invariant.
Let $v \in V$, $T$ a linear tranform $V^T \to V$.

Let $j$ be the smallest integer such that

$T^j(v) \in \text{span} \{v, T(v), T^2(v), \ldots, T^{j-1}(v)\} = B$.

Put $V_{j-1} = \text{span}(B)$, $B$ is clearly a basis for $V_{j-1}$.

$V_{j-1}$ is a subspace of $V_j$ and write $B$,

$$
\begin{bmatrix}
T & V_{j-1} \\
0 & 0
\end{bmatrix}_B
$$

where $T^j(v) = \sum_{i=0}^{j-1} a_i T^i(v)$,

First case:

$v_1 = v \rightarrow T(v) = v_2$.

Now just iterate the process for $V_{j-1}$.

Connecting Linear transforms to K[E] modules:

Theorem (Artin, p476): A linear operator $T$ on a vector $V/k$ and a $K[E]$ module (finitely) are equivalent.

Sketch Pf.

Let $T : V \to V$ a LT, and $f(E) = \sum_{i=0}^{m} a_i E^i$.

We make $V$ into a $K[E]$ module via

$$f(t) \cdot v = \sum_{i=0}^{m} a_i T^i(v)$$

Check that this operator satisfies the module requirements.

- If $V$ is a $K[E]$ module, then $K$ acts on $V$;
- $V$ is a $K$-vector space, and $T$ maps $V \to V$.

The module properties mean it is a linear op.

So define $T = \cdot t$. 

Thus, we have shown

1. $T$ a linear operator on $V$ vsfd over $k$

$$
\begin{array}{cc}
\text{V is a finitely generated } & k[t] \text{ module} \\
\text{remember this is} & \text{a Euclidean Domain!}
\end{array}
$$

2. Any lin. op $T$ on a vsfd over $k$

has a decomposition into a natural canonical form

$$
\begin{pmatrix}
B_{i1} & 0 & 0 \\
0 & B_{i2} & 0 \\
0 & 0 & B_{ik}
\end{pmatrix}
$$

with $B_{ij}$ square invariant blocks

3. Concrete example: Suppose $V = k[t]/(f(t))$, where $f(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_0$. Clearly $V$ is a vsfd over $k[t]$ with basis $1, t, \ldots, t^{n-1}$.

\[ t \in k[t] \ implies \ \begin{pmatrix} t \end{pmatrix} \]

So matrix $A$ is

$$
\begin{pmatrix}
0 & 0 & \ldots & 0 \\
-1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & -1
\end{pmatrix}
$$

invariant!
Jordan Canonical Form

Suppose \( f(t) = (t-\alpha)^n \), \( W = k[t] / \langle f(t) \rangle \).

We use a slightly different basis:
\( w_0 \) := residue class of 1 in \( W \)
\( w_i = (t-\alpha)^i w_0 \)

Then \( (t-\alpha)w_0 = w_i \)
\( (t-\alpha)w_{n-2} = w_{n-1} \) and \( (t-\alpha)w_{n-1} = 0 \)

Hence \( (T-\alpha)w_i = w_{i+1} \) \( \Rightarrow Tw_i = \alpha w_i + w_{i+1} \) \( \{i = 0, n-2\} \)

\( Tw_{n-1} = \alpha w_{n-1} \)

So the matrix for \( T \) is
\[
\begin{bmatrix}
0 & \cdots & 0 \\
1 & 0 & \cdots \\
& 1 & \ddots \\
& & & 0 & \cdots \\
& & & & 1 & \cdots \\
& & & & & & 0 & 1 & \cdots \\
\end{bmatrix}
\]

\( \# \text{ Jordan Block} \)

Notice: If \( k \) is algebraically closed (say, if \( k = \mathbb{C} \))
then any \( f(t) = \prod (t-\alpha_i)^{m_i} \)

So we get a representation of \( Tw \) with Jordan blocks.

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BONUS: Google's PageRank Algorithm

1. Represent the web as a weighted, directed graph
   vertex := website
   edge \( i \rightarrow j \) if site \( i \) has link to site \( j \)

   weight := if site \( i \) has \( k \) links = arrows pointing "out"
   then each arrow out of \( i \)

intuition: If you're at site \( i' \), equally likely to choose one of \( k \) arrows.
Example  
1 links to 3, 4  
2 " 1, 3  
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3 " 4  
4 " 1, 2, 3

(1) From the graph, build a transition matrix:  
\[
T = \begin{bmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
\end{bmatrix}
\]

\(T\) is \(n \times n\), \(n = \#\) websites  
If \(v_m^n\) is \(n \times 1\), \(\sum v_i = 1\),  
then \(T(v_m^n) = v_{m+1}\) = probabilities at time \(m+1\)

(2) But, there is a probability you don't follow a link, but type in a new URL. Assume jumping to any site is equally likely.  
\(\text{DEF } G = (1-p)T + p \left( \frac{1}{n} \mathbf{1} \right)\), \(\mathbf{1}\) is \(n \times n\) matrix with all \(1\)'s  
\(p \in [0, 1]\)  
\(\text{RMK: } G\) is positive + column stochastic  
(all cols sum to 1)

\text{THM (Perron-Frobenius)} A positive, column stochastic matrix has 1 as an eval; corr. vector is all +.

\(\text{THM: } \text{If } V^*\text{ is as above, and } V(0) = \frac{1}{n} \left( \mathbf{1} \right)\), then \)  
\(\lim_{n \to \infty} G^n(V(0)) = V^* \leftarrow \text{the page!} \)

\text{ALG: To find PageRank, solve } Gv = v, \text{ scale } v \text{ so entries sum to 1.}  
\text{Problem: } n \text{ is in the billions!}
Exercises

1. Prove if $v$ vsfd/ke then the cardinality of a basis for $V$ is well defined.  **DEF** $|\text{Basis}| = \dim V$

2. Find the change of basis matrix $A_{B'B} : B \to B'$ for example on P3. (notes)
   
   Show $A_{B'B} \cdot A_{B'B} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3. Find the rational canonical form for $V = K[t]/(t^3 - 4t^2 + 5t - 2) = f(t)$. Hint: factor $f(t)$

4. Diagonalize the matrix $T$ on p8 (notes)