Lecture 3

TDA

Today, we'll wrap up our "Algebra Boot Camp"
- Structure Thm. for f'gend Abelian gps
- modules, associated primes, primary decomp.

Recall that last lecture, we saw that

① Thm (p476, Artin) TFAE
- A linear operator on a k-vector space
- A $k[t]$ module, where k is a field.

Remark (reminder): R a ring, then an R-module M is an Abelian gp under $+$, with R acting linearly on M $r\cdot(m_1+m_2) = rm_1 + rm_2$

② Thm (p479, Artin) Any linear operator T on a VSfd V/k has a matrix where T consists of diagonal blocks which are invariant.

$$B_i = \begin{bmatrix} 0 & 0 & & & a_1 \\ 1 & 0 & & & \vdots \\ 0 & 1 & \ddots & & \\ & 0 & \ddots & 0 & a_{n-1} \\ 0 & 0 & & 1 & a_n \end{bmatrix}$$

$$T = \begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & B_k \end{bmatrix}$$

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Fundamental Thm of Abelian Gps (FTAG)

(11.12 Fraleigh) (A) A f.gen'd Abelian gp A is isomorphic, as a $\mathbb{Z}$-module, to

$$A \simeq \mathbb{Z}^k \bigoplus_{1}^{m} \mathbb{Z}/p_i^{\alpha_i}$$

primes

aside: (p_i not distinct $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \not\simeq \mathbb{Z}/4$)

{ Notice: an abelian gp is a $\mathbb{Z}$-module, because $n \cdot a = \overbrace{a + a + \cdots + a}^{n\text{-times}}$ } { in fact, $\mathbb{Z}$-module $\Leftrightarrow$ Ab. Gp. }

(B) Same hypotheses, then

$$A \simeq \mathbb{Z}^k \bigoplus_{i=1}^{n} \mathbb{Z}/d_i \quad \text{with } d_i \mid d_{i+1}$$

$\boxed{d_i \neq 0, 1}$

Why are (A), (B) same?

<u>Lemma:</u> If p_1, p_2 distinct primes, then

$$\mathbb{Z}/p_1p_2 \simeq \mathbb{Z}/p_1 \times \mathbb{Z}/p_2.$$

//

(p 458 Artin) <u>Thm</u> A an $m \times n$ integer matrix, then $\exists$ P, Q elementary $\mathbb{Z}$-matrices so that

$A' = QAP^{-1}$ is diagonal, of form

$$\left[\begin{array}{ccc|c} d_1 & & & \\ & d_2 & & 0 \\ & & \ddots \; d_n & \\ \hline & 0 & & 0 \end{array}\right]$$

with $d_i > 0$

$d_i \mid d_{i+1}$

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Recall <u>elementary</u> <u>matrices</u>

- add integer multiple of row/col to another ".
- swap 2 rows or 2 cols
- multiply a row/column by a <u>unit</u>

<u>example</u>: add $2r_1$ to r_2

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2a+c & 2b+d \end{bmatrix}$$

<u>Proof</u>: Do elem. ops to get

$$A \to \tilde{A} = \left[\begin{array}{c|c} d_1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right] \quad \star$$

where d_1 divides <u>all</u> entries of B

This is <u>tricky</u>!, key is Euclidean Alg.
To do this:

<u>STEP 1</u> Permute rows/cols to get entry of A with smallest absolute value into position $(1,1)$

<u>STEP 2</u> Use $(1,1)$ entry to clear out row_1, col_1

<u>Warning</u>: This is <u>not</u> as easy as when our matrix is over a field!

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Suppose we have $\begin{bmatrix} 2 & \cdots \\ 3 & \\ \vdots & \\ \vdots & \end{bmatrix}$

Use 2 to reduce 3 to 1. Now go back to STEP 1, moving 1 to (1,1).

STEP 2 (explicitly) $a_{i1} = a_{11} q + r$.

If $r = 0$, a_{i1} is cleared out.
else $r < a_{11}$, go to STEP 1, replacing a_{11} w/ r.

Can't go on forever.

This process gets us to a matrix of the form ☆, but don't have $d_1 \mid$ all entries of B.

STEP 3: If some entry $b \in B$ is not divisible by d_1, add that column to Col_1;
Col_1 now has an entry b.
Repeat step 2.

e.g.

$$\begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \\ \vdots \\ \vdots \end{pmatrix}$$

this is zero. important

$a_{11} = d_1 \nmid b$, so

$b = a_{11} q + r$ $\quad r \neq 0$
$\Rightarrow r < a_{11}$
Back to STEP 1

PROCESS TERMINATES BECAUSE OUR #S KEEP GETTING SMALLER.

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Presentation Matrices

Lemma A f'gend module $M / R \leftarrow$ Noetherian ring has a <u>presentation</u> matrix

$$R^n \xrightarrow{\varphi} R^m \longrightarrow M \longrightarrow 0$$

R^n ↑ relns $\qquad$ R^m ↑ gens (f'gend, OK)

(relns is also f'gend, because submodule of $R^m \leftarrow$ Noetherian)

Clearly a PID is Noetherian (every ideal has 1 gen!)

Aside Can do without Noetherian, just assuming PID

<u>INDUCT</u> on # gens of M.

If 1 gen, then $R \xrightarrow{\varphi} M \to 0$, Kernel is an <u>ideal</u>, but R a pid so $\ker \varphi = \langle f \rangle$

$\Rightarrow$

$$R \xrightarrow{\cdot f} R \to M \to 0 \qquad \text{BASE CASE.}$$

Now suppose true if M has $\leq n$ gens, and take M with $n+1$ gens, $\alpha_1 \ldots \alpha_{n+1}$.

$M/\alpha_1 R$ has n-gens, finite pres (IH)

$\alpha_1 R$ has 1-gen, "

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 & \alpha_1 R \arrow[l] \arrow[d] & R \arrow[l] & R \arrow[l] & 0 \arrow[l] \\
 & M \arrow[d] & & & \\
0 & M/\alpha_1 R \arrow[l] \arrow[d] & R^\alpha \arrow[l] & R^\beta \arrow[l] & 0 \arrow[l] \\
 & 0 & & &
\end{tikzcd}$$

Exercise — use these to get presentation for M.

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Proof (FTAG)

(1) If A is a f'gend Abelian group $=\mathbb{Z}$-module, then we have
$$\mathbb{Z}^m \xrightarrow{\phi} \mathbb{Z}^n \longrightarrow A \longrightarrow 0$$
a presentation, by previous lemma.

(2) By diagonalization thm,
$$\phi = \underbrace{\left[\begin{array}{ccc|c|} d_1 & & 0 & \\ & \ddots & & \\ 0 & & d_k & 0 \\ \hline & 0 & & 0 \end{array}\right]}_{m} \Bigg\} n$$

Notice: This says, since $A = \text{cokernel } \phi$,
that $A \cong \mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \cdots \oplus \mathbb{Z}/d_k \oplus \mathbb{Z}^{n-k}$
with $d_i \mid d_{i+1}$ $\blacksquare$

The same argument works for modules over a PID, see ~~Harts~~ Hungerford, IV. 6. 12

This concludes our work on structure theory for modules over a Euclidean Domain.

Reminder: This structure theory is the fundamental ingredient is the description of PH as a barcode.

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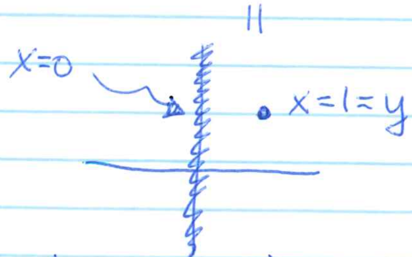
Second Topic of Today: Varieties, Associated Primes, Primary Decomp.

Recall In Lecture 1, For $I \subseteq S = k[x_1, \dots x_n]$ we defined
$$V(I) = \{ p \in k^n \mid f(p) = 0 \ \forall f \in I \}$$

Example $V(x^2 - x, xy - x) \subseteq \mathbb{R}^2$.

$\parallel$



Union of
- a line $V(x)$
- a point $V(x-1, y-1)$.

Notice: $\langle x^2 - x, xy - x \rangle = \langle x \rangle \cap \langle x-1, y-1 \rangle$

Aside: we've not yet met operations on ideals.

Exercise I, J ideals in R, then $\left\{ \begin{array}{l} I + J \\ I \cap J \\ I \cdot J \end{array} \right.$ are ideals in R

DEF: • I an ideal is <u>irreducible</u> if $I \neq J_1 \cap J_2$, $I \subsetneq J_i$
• " primary if $x \cdot y \in I \Rightarrow x \in I$ or $y^m \in I$

Thm R Noetherian, then any $I = \bigcap_i^n I_i$ ← irreducible
(n ← finite)

Pf: Let Σ = set of all ideals which can't be written as above. Since R Noetherian, Σ has a max elt I'. I' is reducible so $I' = I_1 \cap I_2$, $I' \subsetneq I_j$, and I_j <u>are</u> <u>not</u> in Σ

$\Rightarrow I'$ is a finite $\cap$ of irrs, $\Rightarrow\Leftarrow$

DEF $\sqrt{I} = \{ f \mid f^n \in I, \text{ some } n \in \mathbb{Z}_{>0} \}$

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(Schenck, Lem 1.3.3) Fact: In a Noetherian ring, irr $\Rightarrow$ primary.

Hence, in a Noetherian ring

$$I = \cap Q_i \leftarrow \text{primary.}$$

Clearly $\sqrt{Q_i} = P_i$

Example: $I = \langle x^2 - x, xy - x \rangle = \langle x \rangle \cap \langle x-1, y-1 \rangle$

Notice: $\mathbb{R}[x,y]/\langle x \rangle \simeq \mathbb{R}[y]$ Domain $\Rightarrow \langle x \rangle$ prime

$\mathbb{R}[x,y]/\langle x-1, y-1 \rangle \simeq \mathbb{R}$ field $\Rightarrow \langle x-1, y-1 \rangle$ prime

So we've broken $V(I)$ into $\underbrace{V(x) \cup V(x-1, y-1)}_{\text{irr components.}}$

Warning: $\langle x^2, xy \rangle = \langle x^2, y \rangle \cap \langle x \rangle$
$= \langle x^2, xy, y^2 \rangle \cap \langle x \rangle$ } both primary decomps

* Primary Decomposition is not unique
But the $\sqrt{Q_i} = P_i$ are unique.

DEF1 { P_i is an associated prime of I if $I = \cap Q_i$, $\sqrt{Q_i} = P_i$ }

DEF2 M an R-module, P associated prime of M if $P = \operatorname{ann}(m)$, some $m \in M$.

Huh? What?

Notice if P is the annihilator of $m \in M$, then consider the submodule (principle)

$$\begin{array}{ccccc} 0 \to & R \cdot m & \hookrightarrow & M \\ & \uparrow \wr & & \\ & R/P & & \end{array}$$

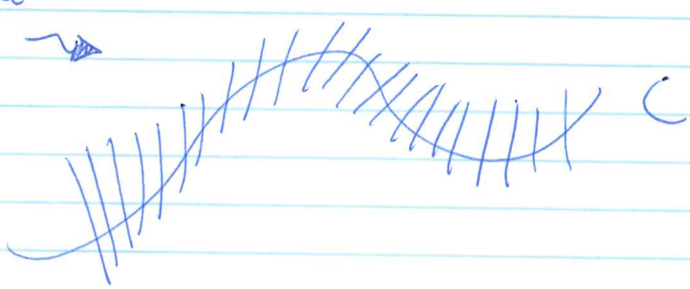
In particular, if $M = R/I$ and $I = \cap Q_i$ with $\sqrt{Q_i} = P_i$

Clearly $R/P_i \hookrightarrow R/Q_i \hookrightarrow R/I$

(since $P_i \supseteq Q_i \supseteq I$) so DEF 1 and DEF 2 agree.

$\boxed{\text{Who cares}}$ Idea: In topology, a vector bundle on a space X is a collection of vector spaces (all $\simeq$) which vary continuously (smoothly, holomorphically, ...) with $p \in X$.

E.g. Line Bundle {1-d vect space} on a curve C "looks like"



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fiber = object over a point. 9

Q: How do we visualize a module?

A: Like a Vector Bundle, but the fibers differ

Key idea: Use localization

Visualize the module $M = R/\langle x^2 - x, xy - x\rangle$ over $\mathbb{R}^2$

Localize M at associated primes

$M_{\langle x\rangle} \leftarrow$ things outside x are units $\Rightarrow x-1, y-1$ units

$\Rightarrow M_{\langle x\rangle} \simeq R/\langle x\rangle \simeq k[y]$ a line

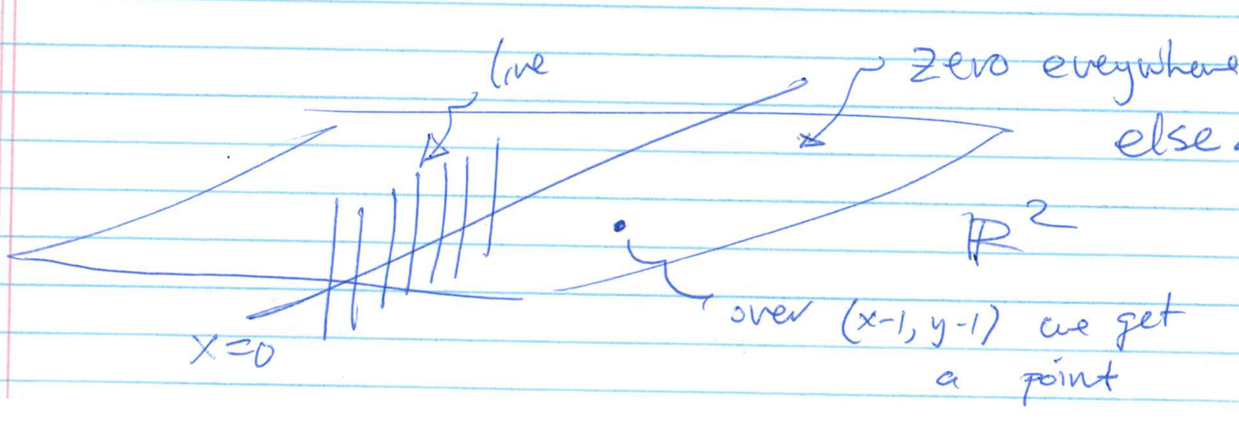
$M_{\langle x-1, y-1\rangle} \Rightarrow x \notin \langle x-1, y-1\rangle \Rightarrow x$ unit $\Rightarrow M_{\langle x-1, y-1\rangle} \simeq R/(x-1, y-1)$

$M_{(p)}$ p a prime $\notin \{\langle x\rangle, \langle x-1, y-1\rangle\}$.

then $x \notin p$, $x-1, y-1 \notin p \Rightarrow$ they are units

$\Rightarrow x(x-1)$ is a unit $\quad R/\text{unit} = 0$

$\boxed{\text{So}}$



Exercises

1. Prove if p_1, p_2 distinct primes, then
$$\mathbb{Z}/_{p_1 p_2} \simeq \mathbb{Z}/_{p_1} \times \mathbb{Z}/_{p_2}.$$

2. Reduce the integer matrix $\begin{bmatrix} 3 & 1 & -4 \\ 2 & -3 & 1 \\ -4 & 6 & -2 \end{bmatrix}$ to diagonal form using (integer) elem. ops.

3. Complete proof on P4 (notes) that a f'gen'd module over a PID has a finite presentation

4. I, J ideals in R, show $\left\{ \begin{matrix} I+J \\ I \cap J \\ I \cdot J \end{matrix} \right.$ are ideals in R