Lecture 4

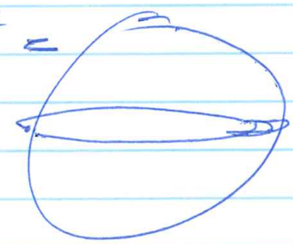
TDA

Next 3 lectures are "Topology Boot Camp"

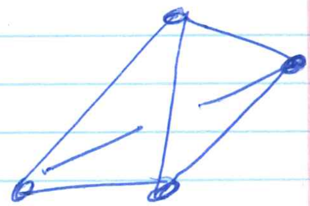
We will, however, use our new topological objects to illustrate some of the objects we encountered in "Algebra Boot Camp".

Basic Idea X a topological space,

e.g. $X = S^2 =$



There is, from a topological standpoint, no difference between S^2 and the boundary of a tetrahedron.



Enrico Betti (Italiano, 1800's) exploited this idea to study surfaces.

Simplices

DEF V a set of $n+1$ elements. The n-simplex on V is the set of all subsets of V.

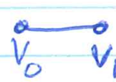
Example

0 simplex

v_0

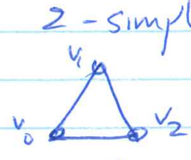
$\{\phi, v_0\}$.

1 simplex



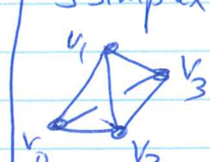
$= \{\phi, v_0, v_1, v_0v_1\}$.

2-simplex



$= \{\phi, v_0, v_1, v_2, v_0v_1, v_0v_2, v_1v_2, v_0v_1v_2\}$

3 simplex



$\vdots$

Example: $V = \vec{0}$ + unit vectors in $\mathbb{R}^n$

"The standard unit n-simplex"

AKA a geometric realization.

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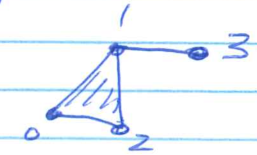
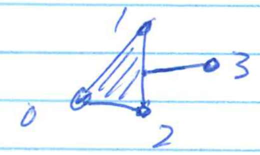
DEF An abstract simplicial complex Δ on "vertex set" V is a collection of subsets of V

- $v_i \in \Delta$ if $v_i \in V$ ← all vertices
- $\theta \subseteq \tau \in \Delta \Rightarrow \theta \in \Delta$ ← if a face $\tau \in \Delta$, all subfaces $\in \Delta$.

$$\dim(\Delta) = \sup\{|\theta| - 1,\ \theta \in \Delta\}$$

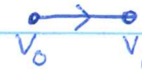
Example: $V = \{0, 1, 2, 3\}$ $\Delta =$ 

$\dim \Delta = 2.$

Nonexample: 

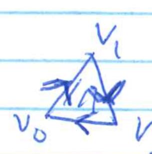
"two faces $\theta, \theta' \in \Delta$ meet at $\theta \cap \theta' = \tau \in \Delta$"

DEF An orientation on Δ is a choice of ordering of the vertices of each $\theta \in \Delta$.

Example 

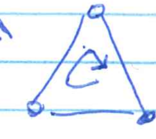
$\Leftrightarrow [v_0, v_1]$

$\Leftrightarrow [v_1, v_0]$



$\Leftrightarrow [v_0, v_1, v_2]$

"



From simplices to algebra = algebraic topology

DEF A sequence of algebraic objects C_i and maps $C_i \xrightarrow{\partial_i} C_{i-1}$ is a chain complex if $\operatorname{im} \partial_i \subseteq \ker \partial_{i-1}$

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In this case, if $C: C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1}$

is a complex, we define $H_i(C_\cdot) = \dfrac{\ker \partial_i}{\operatorname{im} \partial_{i+1}}$.

and say C is exact at position i if $H_i = 0$.

In Exercise 1, you'll show that if

$$V = \quad 0 \to V_n \xrightarrow{d_n} V_{n-1} \to \cdots \to V_1 \xrightarrow{d_1} V_0 \to 0$$

is a complex of vsfd / k, then

$$\sum_{i=0}^{n} (-1)^i \dim V_i = \sum_{i=0}^{n} (-1)^i \dim(H_i(V))$$

This quantity is the <u>Euler characteristic</u> of V

<u>Example</u> $\quad 0 \to \overset{k^3}{V_1} \xrightarrow{d_1} \overset{k^3}{V_0} \xrightarrow{d_0} 0$

$$d_1 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Clearly $\ker d_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
so $\dim H_1 = 1$.

d_1 has rank 2, $\ker d_0 = V_0$, so $\dim H_0 = 1$.

and $\dim H_0 - \dim H_1 = 1 - 1 = 0 = \dim V_0 - \dim V_1$

$\boxed{\text{How in the world does this tie in to topology?}}$

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DEF Let Δ be an oriented simplicial complex, and R a ring (often $R = \mathbb{Z}$ or $\mathbb{Q}$)

$C_i(\Delta)$ = free R-module
w/ basis oriented i-simplices $/\sim$

where $[v_{j_0}, \ldots v_{j_k}] \sim (-1)^{\operatorname{sgn}(\sigma)} [v_{\sigma(j_0)} \ldots v_{\sigma(j_k)}]$

and $\sigma \in S_{k+1} \leftarrow$ permutations of a $k+1$ set.

Example: $[v_0, v_1] \sim -[v_1, v_0]$ makes sense!

$$\underset{v_0}{\bullet}\!\longrightarrow\!\underset{v_1}{\bullet} \;=\; -\left(\underset{v_0}{\bullet}\!\longleftarrow\!\underset{v_1}{\bullet}\right)$$

Remark Technically (in a text on alg. topology)
$c_i \in C_i(\Delta)$ is a map from oriented i-simplices to R, which is 0 except on finitely many simplices, with
$c_i(\delta) = -c_i(\delta')$
if δ, δ' differ by an odd perm.

Because we'll be dealing with finite smp. cpxs, we won't need this.

DEF Define $C_i \xrightarrow{\partial_i} C_{i-1}$ via

$$\partial_i [v_0, \ldots, v_i] = \sum_{j=0}^{i} (-1)^j [v_0, \ldots \hat{v}_j, \ldots v_i]$$

$\hat{v}_j$ = leave it out.

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In Exercise 2, you'll prove

$$\partial^2 = 0, \text{ i.e. } \partial_i \partial_{i+1} = 0$$

Hence, $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$

DEF: • Δ an oriented simplicial cpx, R a ring

$$H_i(\Delta, R) = \ker \partial_i / \operatorname{im} \partial_{i+1}$$

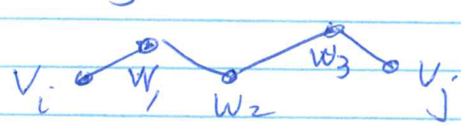
(reduced homology) → • $\widetilde{C}_i(\Delta) = C_i(\Delta) \quad i \geq 0$

$\widetilde{C}_{-1}(\Delta) = R$ (corresponds to empty face)

Question What is this $\widetilde{C}$ stuff? Answer

Lemma If R is a field, $\dim \widetilde{H}_0(\Delta) = \#$ connected components of Δ.
So $\widetilde{H}_0(\Delta) = 0$ when Δ is connected.

Pf: If v_i, v_j lie in same connected comp, then there is a sequence of edges connecting them:



But in homology, $v_i \sim w_1$ $(\partial[w_1, v_i] = w_1 - v_i)$
So " $v_i \sim v_j$

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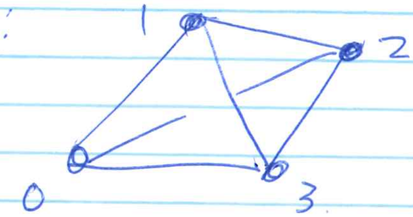
Example:



(Hollow Tetrahedron, $\simeq S^2$)

Compute Homology of above with, say, $\mathbb{Q}$ coeffs.

$$0 \longrightarrow \mathbb{Q}^4 \xrightarrow{\partial_2} \mathbb{Q}^6 \xrightarrow{\partial_1} \mathbb{Q}^4 \longrightarrow 0$$

$$\begin{array}{c|cccc} & 012 & 013 & 023 & 123 \\ \hline 01 & 1 & 1 & & \\ 02 & -1 & & 1 & \\ 03 & & -1 & -1 & \\ 12 & 1 & & & 1 \\ 13 & & 1 & & -1 \\ 23 & & & 1 & 1 \end{array} \qquad \begin{array}{c|cccccc} & 01 & 02 & 03 & 12 & 13 & 23 \\ \hline 0 & -1 & -1 & -1 & & & \\ 1 & 1 & & & -1 & -1 & \\ 2 & & 1 & & 1 & & -1 \\ 3 & & & 1 & & 1 & 1 \end{array}$$

Notice cols of ∂_2 all sum to 1.
easy check: ∂_2 has rank 3
and $\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ $\boxed{\text{gens } \ker \partial_2}$

$\boxed{\dim \operatorname{im} \partial_2 = 3}$

Notice cols of ∂_1 sum to 0
easy check; ∂_1 has rank 3
$\Rightarrow$ $\boxed{\dim \ker \partial_1 = 3}$

Thus, we find $\dim H_2 = 1$

$\dim H_1 = 0$

$\dim H_0 = 1$

Poincaré Duality X a smooth manifold dim n, then $\dim H_i = \dim H_{n-i}$

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When we use $R = \mathbb{Z}$, the <u>ranks</u> of H_i are the Betti #'s.

($H_i(\Delta, \mathbb{Z}) \simeq \mathbb{Z}^r \oplus \mathbb{Z}/d_i$, by FTAG!)

<u>Question</u> Doesn't this depend on <u>triangulation</u>
(triangulation = choice of simp. cpx Δ modelling space)

<u>Answer</u>: No ~ idea is to take, for two simp cpx Δ, Δ', a common refinement.

In Exercise 3, you'll use a different Δ to compute $H_i(S^2, \mathbb{Q})$.

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We'll close this lecture by tying back to algebra, in a <u>really neat way</u>

<u>DEF</u> Δ a simplicial complex on $V = \{v_0 \ldots v_n\}$.
The Stanley-Reisner ring A_Δ is

$$\mathbb{Z}[v_0 \ldots v_n] / I_\Delta$$

I_Δ ← monomial ideal $\Updownarrow$ non faces of Δ.

Huh?

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Ex: For 

, Stanley-Reisner ring is

$$\mathbb{Z}[v_0, v_1, v_2, v_3] \big/ \langle v_0 v_3 ,\ v_2 v_3 \rangle$$

edge 03 is missing
edge 23 is missing.

DEF An ideal $I \subseteq S = K[x_0 \ldots x_n]$ is homogeneous if it can be generated with polys $f = \sum a_i x^{m_i}$, all $|m_i|$ = same (i.e. $x^2 + xy$ homog, $x^2 + y$ not homog)

If I is homogeneous, then S/I is a $\mathbb{Z}$-graded ring, i.e. $S/I = \bigoplus_{j \in \mathbb{Z}} (S/I)_j$

with $\underbrace{r_i}_{\deg i} \cdot \underbrace{v_j}_{\deg j} \in (S/I)_{i+j}$.

Notice: because $(S/I)_0 = k$ constants, each $(S/I)_i$ is a k-vector space

Def: The Hilbert function of a graded ring R is $i \mapsto \dim R_i$

Thm: (Hilbert) R a graded ($\mathbb{Z}$) ring, then for $i \gg 0$, $\dim_k R_i$ is a polynomial $P(i)$ called the Hilbert Poly

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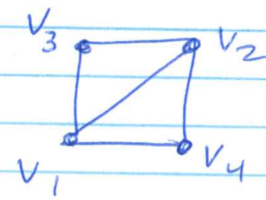
Thm Hilbert R a graded ring, then

$$\sum_i \dim R_i \, t^i \quad \text{is a rational function,}$$

if $R = K[x_0 \ldots x_n]/I$, it is $\dfrac{Q(t)}{(1-t)^{n+1}}$.

[Both thms are an induction on # vars,
See Schenck, Thm 2.3.3, 2.2.4.]

Example $\Delta =$ 

no Δ's
5 edges
4 vertices

$$I_\Delta = \langle \underbrace{v_3 v_4}_{\text{missing edge}},\ \underbrace{v_1 v_2 v_3,\ v_1 v_2 v_4}_{\text{missing } \Delta\text{'s}} \rangle$$

other missing triangles include $v_3 v_4$, not needed.

A check shows $I_\Delta = \langle v_1, v_3 \rangle \cap \langle v_1, v_4 \rangle \cap \langle v_2, v_3 \rangle$
$\cap \langle v_2, v_4 \rangle \cap \langle v_3, v_4 \rangle$

Theorem: $I_\Delta = \displaystyle\bigcap_{v_{i_1} \ldots v_{i_k}} \langle v_{i_1}, \ldots, v_{i_k} \rangle$

minimal coface $\leftarrow$ complement is a face.

(Primary decomp) {check on above}

Thm: The Hilbert polynomial $HP(R/I_\Delta, i)$

$$= \sum_{j=0}^{\dim \Delta} f_j \binom{i-1}{j}, \quad f_j = \# \, j \text{ dim faces.}$$

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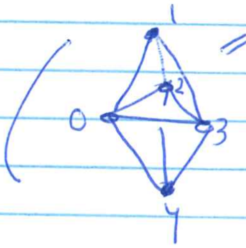
Exercises

1. If $0 \to V_n \xrightarrow{d_n} V_{n-1} \to \cdots \to V_1 \xrightarrow{d_1} V_0 \to 0$ is a complex of vector spaces (vsfd)

   Prove $\sum_{i=0}^{n} (-1)^i \dim V_i = \sum_{i=0}^{n} (-1)^i \dim H_i$

2. Prove $\partial_i \partial_{i+1} = 0$.

3. Compute H_i with $\mathbb{Q}$ coeffs (

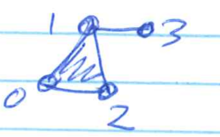


$= \Delta$ Hollow (max Δ's are 012, 013, 123, 024, 234, 034.))

$\boxed{\Delta\ 023 \text{ is not in } \Delta}$

Hint: $\Delta \sim S^2$, so you'll see $\dim H_i = (1,0,1)$.

4. For the Stanley Reisner ring of



, compute

- primary decomp of I_Δ
- Hilb poly of I_Δ.