Lecture 5

TDA

We've done a <u>solid</u> dose of algebra and topology. Today we go back to our motivating problem.

WHY?

- Visual cortex activity
- Breast cancer
- Viral evolution.

① Point cloud data → Filtered top. space
  - Čech complex
  - Vietoris Rips complex

② Formal def. of PH

③ Morse Theory

④ Homology is functorial (categories and functors)

⑤ Stability and interleaving.

§1: In Lecture 1, we had the following setup

$X =$ 

$$X_\varepsilon = \bigcup_{p \in X} N_\varepsilon(p) \qquad X_{\varepsilon'} \subseteq X_\varepsilon \text{ if } \varepsilon' < \varepsilon$$

We have a filtered topological space.

Last Lecture, we learned how to compute simplicial homology: $C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1}(\Delta)$

where Δ was an <u>oriented</u> simplicial complex (and some choice of coefficient ring).

Definition (Čech complex) $\mathcal{C}_\varepsilon$ is a simplicial complex, with k-simplices $\Longleftrightarrow$ $k+1$ tuples of points of X such that

$$\bigcap_{i=1}^{k+1} \overline{N_{\varepsilon/2}(p_i)} \neq \emptyset$$

"The $k+1$ closed balls have a common point"

(Rips (Rips-Vietoris)) complex R_ε has k simplices $\Longleftrightarrow$ $k+1$ sets of points of X, such that $d(p_i, p_j) \leq \varepsilon$ $\forall i, j$

"All points have pairwise distance $\leq \varepsilon$.

Nerve Thm $[\mathcal{C}_\varepsilon$ is homotopy $\sim$ to $X_{\varepsilon/2}]$

$\rightarrow$ Not true for R_ε

However, there is the following nice theorem of **de Silva**: $R_\varepsilon \hookrightarrow \mathcal{C}_{\varepsilon\sqrt{2}} \hookrightarrow \mathcal{R}_{\varepsilon\sqrt{2}}$

"$\mathcal{C}_\varepsilon$ is theoretically better, but computationally very tough to get at, $\mathcal{R}_\varepsilon$ is easy (well, <u>easier</u>) to compute."

Also worth noting is that $\mathcal{R}_\varepsilon$ is entirely determined by <u>vertices</u> <u>and</u> <u>edges</u> (it is

DEF a <u>flag</u> <u>complex</u> $:\Leftrightarrow$ if $\left\{ \begin{array}{l} v_1v_2, v_1v_3, v_2v_3 \in \Delta, \\ v_1v_2v_3 \in \Delta. \end{array} \right\}$)

Aside (Sorry!)

☺ Famous open conjecture Charney-Davis.

If Δ is a flag complex which triangulates S^{2d-1}, then

$$(-1)^d \sum_{i=0}^{2d} (-1)^i \left(\tfrac{1}{2}\right)^i f_{i-1} \geq 0$$

L^2 coho! {Known for d=2 / Davis-Okun}

<u>Persistent Homology</u>

<u>Lemma</u> Δ, Δ' simplicial complexes, $f : \Delta \to \Delta'$ a simplicial map {(i-simplices $\to$ simplices) <u>and</u> another condition}

Then f induces a map on homology.

<u>Alternate</u> $\mathcal{C} = C_i$, $\mathcal{D} = D_i$ two chain complexes, and $f : C_i \to D_i$ such that

$$\begin{tikzcd} C_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f"] & C_i \arrow[r, "\partial_i"] \arrow[d, "f"] & C_{i-1} \arrow[d, "f"] \\ D_{i+1} \arrow[r, "d_{i+1}"] & D_i \arrow[r, "d_i"] & D_{i-1} \end{tikzcd}$$

commutes.

Pf:

$$\begin{tikzcd} \mathcal{C}: C_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1}"] & C_i \arrow[r, "\partial_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[d, "f_{i-1}"] \\ D: D_{i+1} \arrow[r, "d_{i+1}"] & D_i \arrow[r, "d_i"] & D_{i-1} \end{tikzcd}$$

with diagram commuting

$f_{i-1}\partial_i = d_i f_i$

we want to show that $H_i(\mathcal{C}) \to H_i(D)$.

Let $\alpha \in H_i(\mathcal{C})$, so $\partial_i(\alpha) = 0$ (pick a pepresentative, $\alpha \in \ker \partial_i / \operatorname{im} \partial_{i+1}$)

Hence

$$\begin{aligned} f_{i-1}\, &\partial_i(\alpha) = 0 \\ &\| \\ d_i\, &f_i(\alpha) = 0 \\ \Rightarrow\ &f_i(\alpha) \in \ker d_i \to \ker d_i / \operatorname{im} d_{i+1} \end{aligned}$$

Need to check it is well defined -

Suppose $\alpha' = \alpha + \gamma$, $\gamma \in \operatorname{im} \partial_{i+1}$, say $\gamma = \partial_{i+1}(\beta)$.

Then [illegible] $f_i(\alpha') = f_i(\alpha) + \underset{\overset{\|}{~~f_{i+1}\, d_{i+1}(\beta)~~}}{f_i(\partial_{i+1}(\beta))}$

So well defined in $H_i(D)$ $\in \operatorname{im} d_{i+1} \longrightarrow d_{i+1} f_{i+1}(\beta)$

Wait. We were defining PH. What is this?

A: Because $X_{\varepsilon'} \subseteq X_\varepsilon$

$$\left(\text{and } \begin{array}{l} C_{\varepsilon'} \subseteq C_\varepsilon \\ R_{\varepsilon'} \subseteq R_\varepsilon \end{array} \right)$$

we get simplicial maps between the homology

DEF The p^{th} persistent homology of an interval i, j is the image $H_p(K_i) \hookrightarrow H_p(K_j)$ where $K = X, C, R$

AARGH

Rmk: Weinberger defines PH as $\prod_i H_p(K_i)$ ☹

Rmk: $i, j \in \mathbb{R}$, this is still problematic. However, we can discretize: there are only ~~finite~~ discrete pts where X_ε changes:

two pts at distance 1

$X_\varepsilon \simeq X_{\varepsilon'}$ $\forall \varepsilon < \frac{1}{2}$. At $\varepsilon = \frac{1}{2}$, they coalesce.

So, the basic idea is that instead of using our original $\varepsilon \in \mathbb{R}$ parameter which measured our fattenings of the points, we now use $\mathbb{Z}$ (or $\mathbb{N}$), and at each $i \in \mathbb{R}$ where topology changes, we use $\cdot X^i$

In other words, we have an infinite ($\mathbb{Z}$) set

$$\begin{tikzcd} C(i-1) : \cdots \arrow[d, "\cdot X"'] & C(i-1)_{j+1} \arrow[r] & C(i-1)_j \arrow[r] & C(i-1)_{j-1} \\ C(i) : \cdots \arrow[d, dashed, "\cdot X"'] & C(i)_{j+1} \arrow[r] & C(i)_j \arrow[r] & C(i)_{j-1} \\ {} & & & \end{tikzcd}$$

C is the union of all these.

Key is that now $H_*(C)$ will have (if $\mathbb{F}$ is a field) the structure of a $\mathbb{F}[x]$ module. But first, ~~an analogy~~ another way to view PH

§ **MORSE THEORY** As Weinberger (refs) notes, PH can be regarded as a type of *Morse theory*.

DEF M a manifold (locally looks like $\mathbb{R}^n$, patched together)

A smooth fn. $f : M \longrightarrow \mathbb{R}$ is *Morse* if f has a finite # of crit pts $\{f_{x_i}(p)=0, \text{all } i\}$, which are distinct and *nondegenerate*, which means (*2-D case*)

$$H(p)=\begin{bmatrix} f_{xx}(p) & f_{xy}(p) \\ f_{xy}(p) & f_{yy}(p) \end{bmatrix}, \quad \tilde{H}(p)=\det H(p) \text{ is } \neq 0$$

This is a Calculus III flashback —

2nd deriv test; at a critical pt of $z=f(x,y)$

$$\tilde{H}(p) \begin{cases} + \Rightarrow \text{min / max} \\ - \Rightarrow \text{saddle} \\ 0 \Rightarrow \; ? \end{cases}$$

Morse Theory = 2nd deriv test, gone crazy!

Sketch of Basics

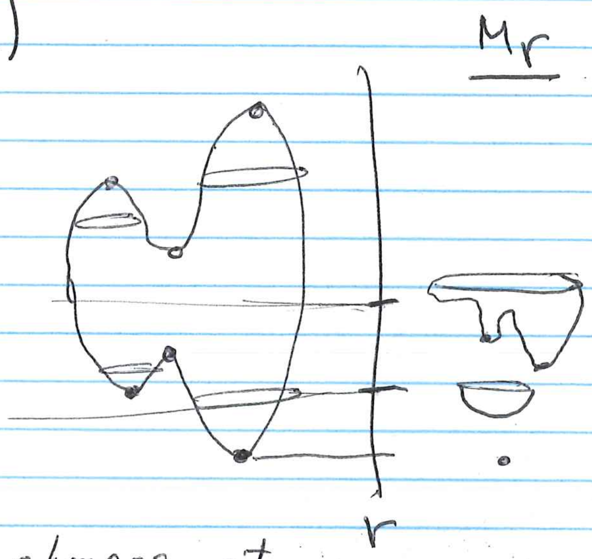
Morse Lemma At a crit pt p, $\exists$ coord system $\{x_1, \ldots, x_n\}$ so that

$$\left[f(\bar{x}) = f(p) + \sum_{i=1}^{\lambda_p} -x_i^2 + \sum_{j=\lambda_p}^{n} x_j^2 \right] \Rightarrow H(p) = \begin{bmatrix} \overbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}^{\lambda_p} & \\ & \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{n-\lambda_p} \end{bmatrix}$$

In particular, λ_p = # negative eigenvals of $H(p)$.

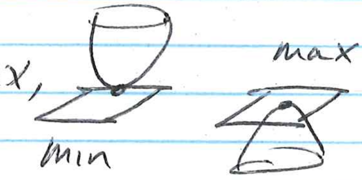
DEF $M_r = f^{-1}((-\infty, r])$

Picture (Weinberger)



Notice: Topology only changes at finite points.

Calc III intuition: At a min/max, we have



Topology changes from "nothing" to "something"!

Morse Thm: • $r < r'$, no critical points in $[r, r']$

Then $M_r \sim M_{r'}$ (hence, same homology)

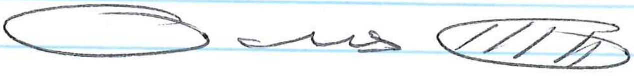
• $r < r'$, r' is the first crit pt $> r$

Then $M_{r'} = M_r + D_{\lambda_p}$, D_{λ_p} a disk of dim λ_p, $\partial(D_{\lambda_p})$ glued to M_r:

__Picture:__ suppose p is a critical pt with $\lambda_p = 2$.

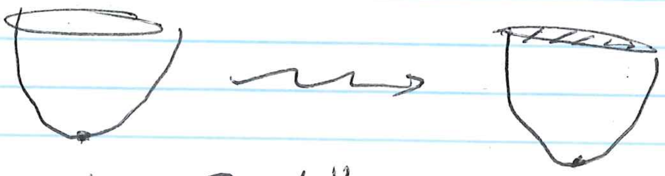
→ we glue a 2-disk to M_r, along the ∂(2-disk) = __circle__

Hence, either



new 2-disk fills a hole $\quad H_1$ __drops__

or



new 2-disk caps off an S^2 $\quad H_2$ __increases__

__Thm:__ At a critical pt p, either $(p' < p)$

First case above ⟹ $\dim H_{\lambda_p - 1}(M_p) = \dim H_{\lambda_p - 1}(M_{p'}) - 1$

OR

Second case ⟹ $\dim H_{\lambda_p}(M_p) = \dim H_{\lambda_p}(M_{p'}) + 1$

__Cor__ Betti #s of M are bounded by

$C_i = \#$ crit pts with $\lambda_p = i$ $\geq \dim H_i(M)$

(Much more, Morse $\neq$) ▤

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⊛ Appendix ⊛

§ Categories + Functors

Another way to define PH ~~is~~ as follows.

? Huh

$\mathcal{C}$ a category, P a poset (cat if $x \to y$ when $x \leq y$)

Persistent object is a functor $P \to \mathcal{C}$

We'll need categories + functors later, so here they are! (but informally)

Def: A category consists of objects and morphisms

Examples Top = { topological spaces, cont. maps }

Diff = { manifolds, smooth maps }.

R-Mod = { modules over a ring R, homs }.

V = { Vector spaces/$\Bbbk$, Linear trans }.

A functor is a map $\mathcal{C}_1 \xrightarrow{F} \mathcal{C}_2$ of categories, taking objects → objects
morph → morph

in a way that "preserves structure"

e.g. • identity → identity
• composition is preserved. etc.

F is covariant if $A_1 \xrightarrow{f} A_2 \Rightarrow F(A_1) \xrightarrow{F(f)} F(A_2)$

and contravariant if $F(A_1) \xleftarrow{F(f)} F(A_2)$

Exercises

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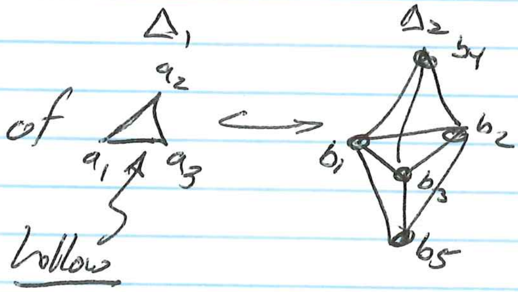
1. a) Compute the Čech and Rips complexes for 4 points at (0,0), (0 1), (1 0), (1,1) and ε values of $\{0, \frac{1}{2}, 1\}$ and the homology.

b) for what ε is C_ε a single 3-simplex?
R_ε " ?

2. For the inclusion of



where the "belt" $\{a_1a_2, a_1a_3, a_2a_3\}$

maps to $\{b_1b_2, b_1b_3, b_2b_3\}$

and Δ_2 is comprised of 6 Δ's (missing Δ's are 123, 145, 245, 345)

find the induced map on homology.

3. Find the critical points and determine the Hessian for the 3 surfaces.

- $z = 2 + x^2 + y^2$
- $z = 2 - x^2 + y^2$
- $z = 2 - x^2 - y^2$

What are the values of λ_p, and what is the picture of the surface?