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Lecture 6

In Lecture 5, we saw that ☆ $\boxed{\text{Homology is Functorial}}$ ☆

that is $\Delta_1 \xrightarrow{f} \Delta_2$

induced a map $H_*(\Delta_1) \xrightarrow{H(f)} H_*(\Delta_2)$.

Today we'll explore more homological algebra; the amazing thing is that abstract <u>algebraic</u> constructions yield beautiful results in <u>topology</u>.

1. Snake Lemma, ses $\to$ les
2. Applications: Mayer-Vietoris, Relative Homology.
3. Chain Homotopy.
4. Cohomology and deRham.
5. Hodge Theory and ranking.

§1 <u>Snake Lemma</u>: <u>DEF</u> Recall that a short exact sequence is a complex

$$0 \to A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} A_3 \to 0$$

which is exact everywhere

$\Longrightarrow$ d_1 an inclusion, d_2 a surjection, image d_1 = kernel d_2.

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Lemma (Snake)

Given 2 s.e.s. and chain map (so all squares commute:)

$$\begin{array}{ccccccccc} 0 & \to & A_1 & \xrightarrow{d_1} & A_2 & \xrightarrow{d_2} & A_3 & \to & 0 \\ & & \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \\ 0 & \to & B_1 & \xrightarrow[\partial_1]{} & B_2 & \xrightarrow[\partial_2]{} & B_3 & \to & 0 \end{array}$$

We have an exact sequence

$$0 \to \text{kernel}(\alpha_1) \to \text{kernel}(\alpha_2) \to \text{kernel}(\alpha_3) \xrightarrow{\delta} \text{cokernel}(\alpha_1) \to \text{cokernel}(\alpha_2) \to \text{coker}(\alpha_3) \to 0.$$

Proof: The hard part is constructing the connecting map δ: $\ker \alpha_3 \to \text{coker}(\alpha_1)$ (recall $\text{coker}\, \alpha_1 = B_1 / \text{im}(\alpha_1)$).

Let $a \in \ker \alpha_3$, so $\alpha_3(a) = 0 \Rightarrow$ (since d_2 is surjective)

$\exists\, b \in A_2$ with $d_2 b = a$

$\Rightarrow \alpha_3 d_2 (b) = 0$

$\Updownarrow$ (square commutes

$\partial_2 \alpha_2 (b) = 0$

But now, $\partial_2 \alpha_2 (b) = 0$

$$\Rightarrow \alpha_2(b) \in \ker \partial_2 \overset{\text{(exact)}}{=} \operatorname{im} \partial_1$$

$$\Rightarrow \exists\, c \in B_1 \text{ with}$$

$$\partial_1(c) = \alpha_2(b).$$

$$\boxed{\underline{\text{Define}} \quad \delta(a) = c \in B_1 \longrightarrow B_1/\operatorname{im}(\alpha_1).}$$

Here!

Well defined? Where did we make choices? (1) $\exists\, b \in A_2 \quad d_2(b) = a$

Suppose $d_2(b') = a \Rightarrow d_2(b - b') = 0$

$$\Rightarrow b - b' \in \ker d_2 = \operatorname{im} d_1$$

$$\Rightarrow b - b' = d_1(e).$$

$$\Rightarrow \alpha_2(b - b') = \alpha_2 d_1(e) = \partial_1(\alpha_1(e))$$

~~$\Rightarrow \exists\, f \in B_1$ with $\partial_1(f) = \alpha_2(b) - \alpha_2(b')$~~

~~$\Rightarrow \alpha_2(b) = \alpha_2(b') + \partial_1(f)$~~

~~$\Rightarrow$ in B_1~~

but ∂_1 is an inclusion, so

$$\partial_1^{-1}(\alpha_2(b - b')) = \alpha_1(e), \text{ i.e.,}$$

the pullbacks of $\alpha_2 b$, $\alpha_2 b'$ differ by an image $\alpha_1(e)$, which means $\boxed{\text{same}}$ in $B_1/\operatorname{im}(\alpha_1)$ ∎

Where is snake?

$\ker(\alpha_1) \to \ker(\alpha_2) \to \ker(\alpha_3)$

$0 \to A_1 \to A_2 \to A_3 \to$

$0 \to B_1 \to B_2 \to B_3 \to 0$

$\operatorname{coker} \alpha_1 \to \operatorname{coker}(\alpha_2) \to \operatorname{coker} \alpha_3$

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§ Fundamental Thm of Homological Algebra (FTHA)

<u>DEF</u>: A s.e.s of complexes is a commutative diagram where <u>rows</u> are <u>complexes</u> and columns are exact:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & & \\
A_\bullet: & 0 & A_0 \arrow[l] \arrow[d] & A_1 \arrow[l] \arrow[d] & A_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & A_n \arrow[l] & 0 \arrow[l] \\
B_\bullet: & 0 & B_0 \arrow[l] \arrow[d] & B_1 \arrow[l] \arrow[d] & B_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & B_n \arrow[l] & 0 \arrow[l] \\
C_\bullet: & 0 & C_0 \arrow[l] \arrow[d] & C_1 \arrow[l] \arrow[d] & C_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & C_n \arrow[l] & 0 \arrow[l] \\
 & & 0 & 0 & 0 & & &
\end{tikzcd}
$$

complex.

exact

<u>THM</u> A s.e.s of complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ gives a les (long exact sequence) in homology:

$$0 \leftarrow H_0(C) \leftarrow H_0 B \leftarrow H_0(A) \leftarrow H_1(C) \leftarrow H_1(B) \leftarrow H_1(A) \leftarrow H_2(C) -$$

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Proof: induct on n. Base case is Snake Lemma!

Only $\overset{\wedge}{\text{mildly}}$ hard part is gluing things together. (Exercise) ☺

Application 1 Mayer-Vietoris sequence.

X, Y simplicial complexes. How do the Homologies $X, Y, X\cup Y, X\cap Y$ relate?

First, notice that we have

$$0 \to C_*(X\cap Y) \to \begin{matrix} C_*(X) \\ \oplus \\ C_*(Y) \end{matrix} \to C_*(X\cup Y) \to 0$$

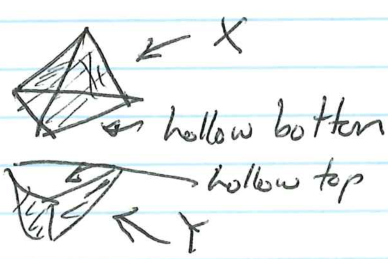
Via the map. $(\sigma_1, \sigma_2) \to \sigma_1 + \sigma_2$

and $(\sigma_1, \sigma_2) \mapsto 0 \Leftrightarrow \sigma_1 = -\sigma_2$
$\Rightarrow$ in $X \cap Y$.

Immediate we get a les in homology

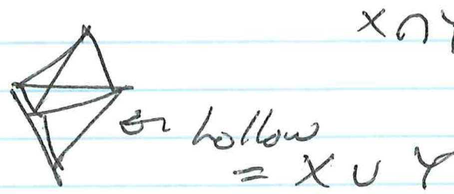
$$\cdots \to H_{i+1}(X\cup Y) \to H_i(X\cap Y) \to \begin{matrix} H_i(X) \\ \oplus \\ H_i(Y) \end{matrix} \to H_i(X\cup Y) \to \cdots$$

Example



glued along

and



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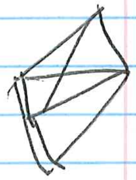
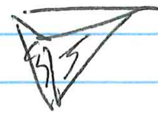
We get, since:  has $C_2=0$, $C_1=R^3$, $C_0=R^3$,

 has $C_2=R^6$, $C_1=R^9$, $C_0=R^5$

 has $C_2=R^3$, $C_1=R^6$, $C_0=R^7$

a ses chain complexes

$$\begin{array}{lccccccccc}
\cap: & 0 & \leftarrow & R^3 & \leftarrow & R^3 & \leftarrow & 0 & \leftarrow & 0 \\
 & & & \downarrow & & \downarrow & & & & \\
\oplus: & 0 & \leftarrow & R^8 & \leftarrow & R^{12} & \leftarrow & R^6 & \leftarrow & 0 \\
 & & & \downarrow & & \downarrow & & & & \\
\cup: & 0 & \leftarrow & R^5 & \leftarrow & R^9 & \leftarrow & R^6 & \leftarrow & 0 \\
 & & & \text{verts} & & \text{edges} & & \Delta s & &
\end{array}$$

Application 2 Relative Homology. More generally, if $\Delta' \subseteq \Delta$, we get

$0 \to C_i(\Delta') \hookrightarrow C_i(\Delta)$. Define the cokernel as

$$C_i(\Delta, \Delta') = \frac{C_i(\Delta)}{C_i(\Delta')} \to 0$$

We get a les in homology.

$H_i(\Delta, \Delta')$ is the relative homology

---

§ Chain Homotopy.

DEF Let A, B be chain complexes, and α_i, β_i chain maps

$$\begin{tikzcd}
A_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, shift right, "\alpha_{i+1}"'] \arrow[d, shift left, "\beta_{i+1}"] & A_i \arrow[r, "d_i"] & A_{i-1} \\
B_{i+1} \arrow[r, "\partial_{i+1}"] & B_i \arrow[r, "\partial_i"] & B_{i-1}
\end{tikzcd}$$

...

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If there exists a family of homomorphisms $\gamma_i : A_i \longrightarrow B_{i+1}$ such that

$$\alpha - \beta = \gamma d + \partial \gamma$$

then α and β are <u>homotopic</u>.

THM if $\alpha \sim \beta$, then they induce the same map on homology.

Pf: We show $\alpha - \beta$ induces the zero map.

Let $c \in H_i(A)$, $c \in A_i$

$(\alpha - \beta)(c)$
$= (\gamma d + \partial \gamma)(c)$

$$\begin{tikzcd}
A_{i+1} \arrow[r] & A_i \arrow[r, "d"] \arrow[dl, "\gamma"'] & A_{i-1} \arrow[dl, "\gamma"'] \\
B_{i+1} \arrow[r, "\partial"] & B_i &
\end{tikzcd}$$

First, $c \in H_i(A) \Rightarrow d(c) = 0$ so $\gamma(d(c)) = 0$.

Second, the image $\partial\gamma(c) \in \operatorname{im} \partial_{i+1} = 0$ in $H_i(B)$.

So $(\gamma d + \partial \gamma)(c) = 0$ in $H_i(B)$ ∎

We'll care about this often when trying to show some homology is <u>zero</u>.

(($\alpha = id$, $\beta = 0$, $\alpha \sim \beta \Rightarrow H_i = 0$))

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§ Cohomology and de Rham.

Define $C^i(\Delta) = \mathrm{Hom}_R(C_i(\Delta), R)$.

When working with R a <u>field</u>, same vector spaces.

But if $R = \mathbb{Z}$, Note $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}) \simeq \mathbb{Z}$, not $\mathbb{Z} \oplus \mathbb{Z}/2$.

Interestingly, $C_i(\Delta)$ does not have a <u>ring structure</u>

<u>Def</u>: $C^i(\Delta) \times C^j(\Delta) \xrightarrow{\cup} C^{i+j}(\Delta)$

via $(c_i, c_j) \in C^i \times C^j$ <u>acts</u> on (it is a <u>Hom</u>) an element of $C_{i+j}(\Delta)$:

Let $\sigma = [v_0 \ldots v_{i+j}] \in$ ↗. R is a ring

Then $(c_i, c_j)(\sigma) = c_i[v_0 \ldots v_i] \cdot c_j[v_i \ldots v_{i+j}]$.

Seems very artificial or contrived. But it connects (another Calc II flashback) to

<u>DEF</u> de Rham cohomology: M a smooth manifold,

$\Omega^k(M)$ = sheaf {a <u>functor</u>, assigns to an open set $U \subseteq M$ an algebraic object} of differential k-forms.

differentiation

$\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)$

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<u>Rough idea</u>: on a small enough open set U,

M "looks like" $\mathbb{R}^n$, which has coordinates $x_1 \ldots x_n$.

<u>Then</u> a 0 form $\in \Omega^0 M(U)$ = function

1 fun $\in \Omega^1 M(U) = \sum_1^n f_i \, dx_i$

2 fun $\in \Omega^2 M(U) = \sum_{i<j} f_{ij} \, dx_i \wedge dx_j$

And d just takes $\sum f_\gamma \, dx_\gamma \mapsto \sum df_\gamma \wedge dx_\gamma$

e.g. $f \mapsto \underbrace{\sum \frac{\partial f}{\partial x_i} dx_i}_{df}$ etc. <u>Not so bad!</u>

<u>Exercise</u> $d^2 = 0$.

<u>Def</u>: $H^i_{dR}(X) = H^i(0 \to \Omega^0 M \to \Omega^1(M) \to \ldots)$

Notice — this has a multiplicative (cup product) structure.

//

Who Cares?

$\boxed{\text{Back to something Concrete}}$ (Lin Alg versions of this)

(Thm) <u>Hodge</u>: In the setting above, there is an <u>adjoint</u> $\Omega^{k+1} \xrightarrow{(d^k)^*} \Omega^k$,

and $\Omega^k \simeq \operatorname{im} d^{k-1} \oplus \operatorname{im}(d^k)^* \oplus \operatorname{Ker} L$,

where $L = (d^k)^* d^k + d^{k-1} (d^{k-1})^*$.

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L = Laplacian, $\ker L$ = Harmonic Forms.

DEF: V, W inner product spaces, $V \xrightarrow{T} W$.
Then T^* the adjoint satisfies $\langle Tv, w\rangle = \langle v, T^* w\rangle$.

{V, w Col vects}

Rmk: Choosing bases so T is a matrix, $T^* = T^t$
$$\left(\left(\langle Tv, w\rangle = w^t \cdot Tv = (Tv)^t \cdot w = v^t \cdot T^t w = \langle v, T^* w\rangle\right)\right)$$

Key in "honest" proof is the Hodge star operator.
[Vector space w/ basis "pf"] (sketch).

$$V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} V_3 \quad \text{a complex,}$$

$\dim V_i = a_i$
$\operatorname{rk} d_i = r_i$
$\dim \ker d_i = k_i = a_i - r_i$.

We can choose bases so
$$d_1 = \begin{array}{c} \overbrace{\begin{array}{cc} r_1 & k_1 \end{array}}^{a_1} \\ \left[\begin{array}{c|c} I_{r_1} & 0 \\ \hline 0 & 0 \end{array}\right] \end{array} \Big\} a_2 \qquad d_2 = \begin{array}{c} \begin{array}{cc} k_2 & r_2 \end{array} \\ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{r_2} \end{array}\right] \end{array}$$

Then $d_1 d_1^t$ and $d_2^t d_2$ are both $a_2 \times a_2$ matrices

$$\left[\begin{array}{c|c} I_{r_1} & \\ \hline & \end{array}\right] \qquad \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{r_2} \end{array}\right] \Rightarrow d_1 d_1^t + d_2^t d_2 = \begin{array}{c} \begin{array}{ccc} & k_2 - r_1 & \end{array} \\ \left[\begin{array}{c|c|c} I_{r_1} & 0 & 0 \\ \hline 0 & \boxed{0} & 0 \\ \hline 0 & 0 & I_{r_2} \end{array}\right] \end{array}$$

and $V_2 \cong \operatorname{im} d_1 \oplus \operatorname{im} d_2^*$
$\oplus \ker L$

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So what? This gives us a canonical set of generators for homology.

$\boxed{\text{Application To Ranking}}$ Jiang-Lim-Yao-Ye

create a weighted directed graph G { Voters rank k movies, they can rank cyclically $a>b>c>d>a$.

Aggregate this data to get global ranking.

THM (JLYY): Think of assignment of weights to an edge of Δ_G {oriented 1-d simp cpx!} as a linear functional $\Rightarrow C_1(\Delta_G)^*$

Let Δ_G be the 2-D simp cpx where if a Δ has all edges in G, we fill it in. (flag)

$C_1(\Delta_G) = \operatorname{im} d_2 \oplus \operatorname{im} d_1^* \oplus \ker L$, where

(1) $\operatorname{im} d_2$ = locally inconsistent rankings $v_i > v_j > v_k > v_i$ (3-cycle)

(2) $\ker L$ = globally inconsistent rankings $v_i > v_j > v_k > \cdots > v_i$ (long cycle)

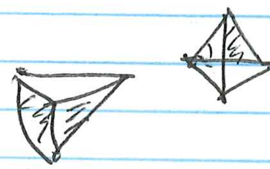
(3) $\operatorname{im}(d_1^*)$ = consistent rankings (no cycle).

Best consistent rank $\Rightarrow$ project onto $\operatorname{im}(d_1^*)$ "Hodge Rank".

Exercises

1. Fully prove the snake Lemma. Already did the hard part = show δ is well defined.

2. Use the snake lemma to show ses (complexes) $\Rightarrow$ les in homology.

3. For the example  glued along 
Compute the homology.

4. Let E be the exterior algebra on basis $e_1 \ldots e_n$ $\longleftarrow$ $V \cong K^n$, K a field. Recall this means $E = \bigoplus_{i=0}^{n} \Lambda^i V$, and $\Lambda^i V$ is a vector space w/ basis $e_{j_1} \wedge \ldots \wedge e_{j_i}$.

$\boxed{\text{AND}}$ relations $\underset{\substack{\uparrow \\ m \text{ tuple}}}{a} \wedge \underset{\substack{\uparrow \\ n \text{ tuple}}}{b} = (-1)^{mn}\, b \wedge a$.

For $n = 3$, write down the parts $\underbrace{E_0, E_1, E_2, E_3}_{\text{dimensions}}$ and multiplication table.
$(1, 3, 3, 1)$

* Now Show if instead $R = K[x_1 \ldots x_n]$, then if $V = R^n$ with basis $dx_1 \ldots dx_n$, we have a differential $\Lambda^i R^n \longrightarrow \Lambda^{i+1} R^n$ as in P.8 of notes.