Lecture 7

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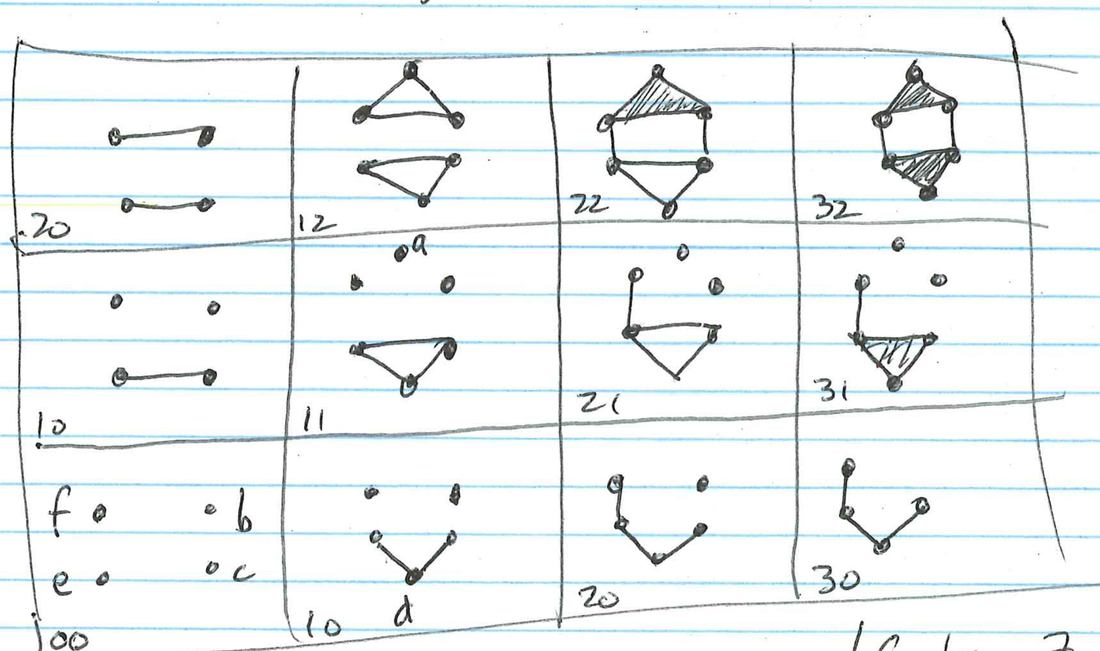
Today, we introduce multiparameter P.H.

In contrast to PH, where we have modules over $\mathbb{F}[x]$ and the corresponding Structure Thm $\Leftrightarrow$ Bar code

now we have modules over $\mathbb{F}[x_1, \ldots x_n]$. $= \left\{ \begin{array}{l} \text{Algebraic} \\ \text{Geometry} \end{array} \right.$

0. MPH examples and definitions
1. Sheaves
2. Review of Assoc. primes, primary decomp.
3. Hilbert function, poly, series
4. Examples + Main Result.
5. Local cohomology and (start) derived functors.

ON BOARD



A Bifiltration (Carlsson - Zomorodian J. Comp Geom 2010)

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tris, edges, verts

We have $\triangle$, $\bullet\!-\!\bullet$, $\bullet$

New twist – they appear at a specific "time" in $\mathbb{Z}^2$

(aside – Carlsson - Zomorodian call a bifiltration 1-critical if $\odot$ appears once, if not, there's a trick to get to a 1-crit filtration)

Key: our differentials will encode "time".

$\boxed{\text{Example 1}}$ edge $\overline{ab}$ appears at $(1,2)$. $x_1\ x_2$ ← weight in 1st, 2nd "time" coord.

a appears at $(1,1)$
b appears at $(0,0)$.

$$\underset{12}{\partial(\overline{ab})} = \underset{00}{b} - \underset{11}{a}. \quad \text{almost}$$

So $\partial[\overline{ab}] = x_1 x_2^2 [b] - x_2 [a]$

$\partial_1 =$

	ab	af	bc	bf	cd	ce	de	ef
a	$-x_2$	$-x_2$						
b	$x_1 x_2^2$		$-x_1^2 x_2^2$	$-x_2^2$				
c			$x_1^2 x_2^2$		$-x_1$	$-x_2$		
d					1		-1	
e						x_2	x_1	$-x_1^2$
f		$x_1 x_2^2$		x_2^2				x_1^2

$\partial_2 =$

	abf	cde
ab	$-x_1$	
af	x_1	
bc		
bf	$-x_1^2$	
cd		$-x_1^2 x_2$
ce		x_1^3
de		$-x_1^2 x_2$
ef		

Example easy check shows

$$\gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \hline -x_2 \\ x_1 \\ -x_2 \\ 0 \end{pmatrix} \in \ker d_1 \text{, and } x_1^2 \cdot \gamma = \partial(cde) \quad \{\text{second col. of } d_2\}$$

So in H_2 we have $\gamma \simeq R/x_1^2$.

Gradings: Poly ring $\underbrace{k[x_1, x_2]}_{=R}$ is graded,

both by $\mathbb{Z}$: bases

			dim
R_0	basis	1 (k-vect sp.)	1
R_1	"	$\langle x_1, x_2 \rangle$	2
R_2	"	$\langle x_1^2, x_1 x_2, x_2^2 \rangle$	3
			⋮

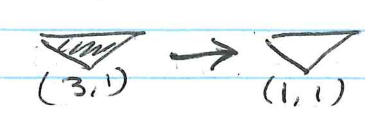
$R(-a) = R$, but with 1 in degree a.

$$\boxed{\text{So } R(-a) \xrightarrow{\text{of degree } a} R}$$

Finer info: want to preserve the grading

In our example, $H_1 \simeq R(-2,-2) \oplus \overline{R(-1,-1)}/x_1^2 \oplus \overline{R(-1,-2)}/x_1$

because



grading shift

Idea: the (x_1, x_2) information will record birth / death.

But now there is more: over $\mathbb{F}[x]$, $\left\{ \overset{\text{long bar}}{M} \oplus \overset{\text{short bar}}{M/x_i} \right\}$

$\mathbb{F}[x] = M$

Now we have more interesting information

《Carlsson – Zomorodian "We still desire a discriminating invariant that captures persistent information"》

Answer: Primary decomposition + Associated Primes

§2 Sheaves

The way to visualize a module is as a sheaf. [Turbo version!]

Formal def: X a top. space, a sheaf $\mathcal{F}$ is a functor $(\underset{}{\overset{\text{open sets of}}{X}},\ \text{inclusion maps}) \xrightarrow{\mathcal{F}} (\text{Alg obj}, \text{Morph})$ with properties that insure gluing

U_i a cover of U

— $s \in \mathcal{F}(U)$ has $s|_{U_i} = 0 \Rightarrow s = 0$ in $\mathcal{F}(U)$

— $s_i \in \mathcal{F}(U_i)$, $s_j \in \mathcal{F}(U_j)$ agree on $U_i \cap U_j$

$$\left.\begin{array}{l} \text{then } \exists\ s \in \mathcal{F}(U_i \cup U_j) \text{ s.t.} \\ s|_{U_i} = s_i \end{array}\right\} \text{gluing}$$

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Now, it seems backwards, but we R.
can go from a module M on $\mathbb{F}[x_1..x_n]$
to a sheaf. In alg geom, we
use the Zarcki top: Varieties are closed
sets.

Def $U_f = V(f)^c$ open set

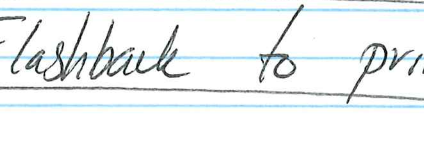
M an R-module, then $M(U_f) = M_{\langle f \rangle}$ ← localization
and for a prime ideal = closed set, the
stalk {think of as the fiber} is M_p.

Holy Moly! What did all that mean?
Point is to "see" M a module on $\mathbb{F}[x_1..x_n]$
we consider where $M_p \neq 0 \iff p \in Ass(M)$.
[Flashback to primary decomp.]

$R = \mathbb{R}[x,y]$.

An example will help. Suppose $M = R/x \oplus R/y$

Then $M_p \neq 0 \iff p = \langle x \rangle$ or $p = \langle y \rangle$ or $p = \langle x, y \rangle$



$M_p \neq 0$ only over coord axes!

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§ Hilbert Function, Series, poly. (Graded ring / module)

Recall $I \subseteq R = k[x_1 .. x_s]$ is <u>homogeneous</u>

if $I = \langle f_1 .. f_m \rangle$ where each f_i is a sum of <u>monomials</u> of same degree.

e.g. $k[x,y] / \langle x^2, xy \rangle = R$

graded pieces of R_i

degree	0	1	2	3	---
basis:	1	x y	y^2	y^3	---

Hilbert Function = dim 1 2 1 1 . ---

Hilbert Poly = dim, if degree $\gg 0$ $\boxed{1}$

Hilbert Series : $\sum HF(i) t^i$

$1 + 2t + t^2 + t^3 + \cdots$

Can do the same not just for $\mathbb{Z}$-graded, but for $\mathbb{Z}^2$ graded, i.e

in $k[x,y]$, x = degree $(1,0)$

y = degree $(0,1)$

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Then we get a bigraded HF

$$HF(M, (a,b)) = \dim_k M_{(a,b)}$$

bigraded Hilb series $\displaystyle\sum_{\mathbb{Z}^2} \dim M_{(a,b)}\, t_1^a t_2^b$

In our example, For H_1 we get

$\uparrow x_1$

↑	↑	↑	↑	↑	↑	
0	0	1	1	1	1	→
0	0	1	1	1	1	→
0	1	2	2	2	2	→
0	1	2	2	2	2	→
0	0	0				

$x_2 \rightarrow$

=

		↑	↑	
0	0	1	1	→
0	0	1	1	→
0	0	0	0	→
0	0	0	0	→

$= R(-2,-2)$

Leaving

0	0	0	0	0	→
0	1	1	1	1	→
0	1	2	2	2	→
0	0	0	0	0	→

+

=

0	0	0	0	0
0	1	1	1	1
0	1	1	1	1
0	0	0	0	0

$R(-1,-1)$

x_1^2

$\oplus$

0	0	0	0	0	…
0	0	1	1	1	…
0	0	0	0	0	…

$R(-1,-2)$

x_1

(B:) Multigraded Hilb series

is
$$\frac{T_1T_2 + T_1T_2^2 - T_1^3T_2}{(1-T_1)(1-T_2)}$$

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$\boxed{\text{So what?}}$

Key point is that the maps in MPH (and modules) are $\mathbb{Z}^r$ graded, $r =$ dim of filtration.

<u>Question</u>: What is a homogeneous ideal in $\mathbb{Z}^2$ grading? What is a poly of bidegree (a,b)? A: $x_1^a x_2^b$ <u>is it</u>.

So only <u>homogeneous</u> $\mathbb{Z}^r$ ideals are monomial ideals

$\{M$ - finitely gen'd graded $R = k[x_1, \dots, x_n]$ module$\}$

<u>Exercise</u> The only prime monomial ideals are subsets of the variables.

[<u>Recall</u> $p \in Ass(M)$ if $P = \operatorname{ann}(m)$, some $\in M$.

easy exercise $p \in Ass(M) \iff M_p \neq 0$.]

<u>Cor</u>: The associated primes of an MPH module are of the form $\langle x_{i_1}, \dots, x_{i_k} \rangle$

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In plain english: an MPH module is supported along coordinate subspaces. (including possibly the whole space).

<u>DEF</u> R an integral domain. The <u>rank</u> of a f.gen'd R-module M is

$$\dim_{R_0} M_0, \text{ where } R_0 = R \text{ localized at } 0 = \text{Field of fractions}$$

(like $\mathbb{Z} \to \mathbb{Q}$).

Beautiful Fact: If Multigraded HS is

$$\frac{P(t_1 \ldots t_r)}{\prod_{1}^{r}(1-t_i)}, \text{ then rank } M = P(1, \ldots, 1)$$

In our example, Multigraded HS is

$$P(t_1, t_2) \longrightarrow \frac{t_1 t_2 + t_1 t_2^2 - t_1^3 t_2}{(1-t_1)(1-t_2)}$$

and $P(1,1) = \boxed{1}$

This is the MPH analog of the # of long bars!

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§ **Look Ahead:** Local cohomology, Ext.

For a bigraded MPH module, due to the $\mathbb{Z}^2$ grading, our associated primes are

$$\{\, \underbrace{0}_{\substack{\text{2d component} \\ \text{of MPH} \\ \text{"long bar"}}},\ \underbrace{\langle x_1\rangle, \langle x_2\rangle}_{\substack{\text{1-d component} \\ \text{live forever,} \\ \text{not full dimensional}}},\ \underbrace{\langle x_1, x_2\rangle}_{\substack{\text{0-d component} \\ \text{"short bar"}}} \,\}$$

DEF $H^0_I(M) = \{ m \in M \mid I^n m = 0, \text{ some } n \gg 0 \}$.

I: ideal in R; M: R-module

Local cohomology gives a way to stratify MPH. How to compute? A: (next time) Local duality and Ext ← derived functor.

Example: $R = K[x,y]$, $M = R/(x^2, xy)$.

Step 1: take a free res of $M \leftarrow R \xleftarrow{[x^2,\, xy]} R(-2)^2 \xleftarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R(-3) \leftarrow 0$

Step 2: drop M + apply $\mathrm{Hom}(-,R)$

$$0 \to R \xrightarrow{\begin{pmatrix} x^2 \\ xy \end{pmatrix}} R^2(2) \xrightarrow{(y,\, -x)} R(3) \to 0$$

Step 3: $\mathrm{Ext}^i(M,R) = H_i$ of this

Exercise: Do this!

Exercises

① For Example 1, we computed H_1. Compute H_2. Don't forget the grading

② <u>Exercise</u> determine which monomial ideals are prime, primary, radical (give proof)

③ Compute homology of

$$
\begin{array}{ccccccccc}
i = & \longrightarrow & 0 & & 1 & & 2. & & \\
0 \to & R & \xrightarrow[\binom{x^2}{xy}]{} & R^2(2) & \xrightarrow[(-y,\,x)]{} & R(3) & \to 0
\end{array}
$$

Hint / example : $H_2 = Ext^2(M,R)$ = homology here

$R(3) \to 0$ so Kernel = all R

mod image of $\langle -y, x \rangle$

$$\Rightarrow Ext^2(M,R) \cong \frac{R(3)}{\langle y, x \rangle} \quad \blacksquare$$