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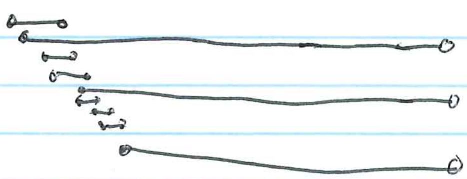
Lecture 8

Last time: We saw that MPH arises from a filtered simplicial complex; when the filtration depends on a single parameter we get "classical" PH, represented as a barcode



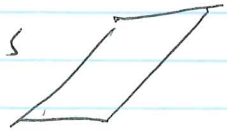
$R = \mathbb{F}[x]$, $\mathbb{F}$ a field

$$\Longleftrightarrow \underbrace{\bigoplus_{i=1}^{3} x^{\alpha_i}\mathbb{F}[x]}_{\substack{\text{long bars} \\ \text{free summands}}} \bigoplus \underbrace{\bigoplus_{j=1}^{6} x^{\beta_j}\mathbb{F}[x]/x^{\gamma_j}}_{\substack{\text{short bars} \\ \text{torsion}}}$$

α_i, β_j birth times,
γ_k = death times.

Key observation is that now $R = \mathbb{F}[x_1, \dots, x_n]$
n = # parameters (directions) of the filtration
gave MPH modules with a $\mathbb{Z}^n$ grading, leading
[to Thm: MPH module is supported on a union of coordinate subspaces]

i.e. MPH module "looks like" $(n=2)$

some (fattened) planes 

some (") lines (coord) 

som ( ) origins $x=y=0 \rightarrow$ 

0. MPH example
1. Derived Functors
2. Spectral Sequences
3. Applications
4. Appendix: Grothendieck spec. seq.

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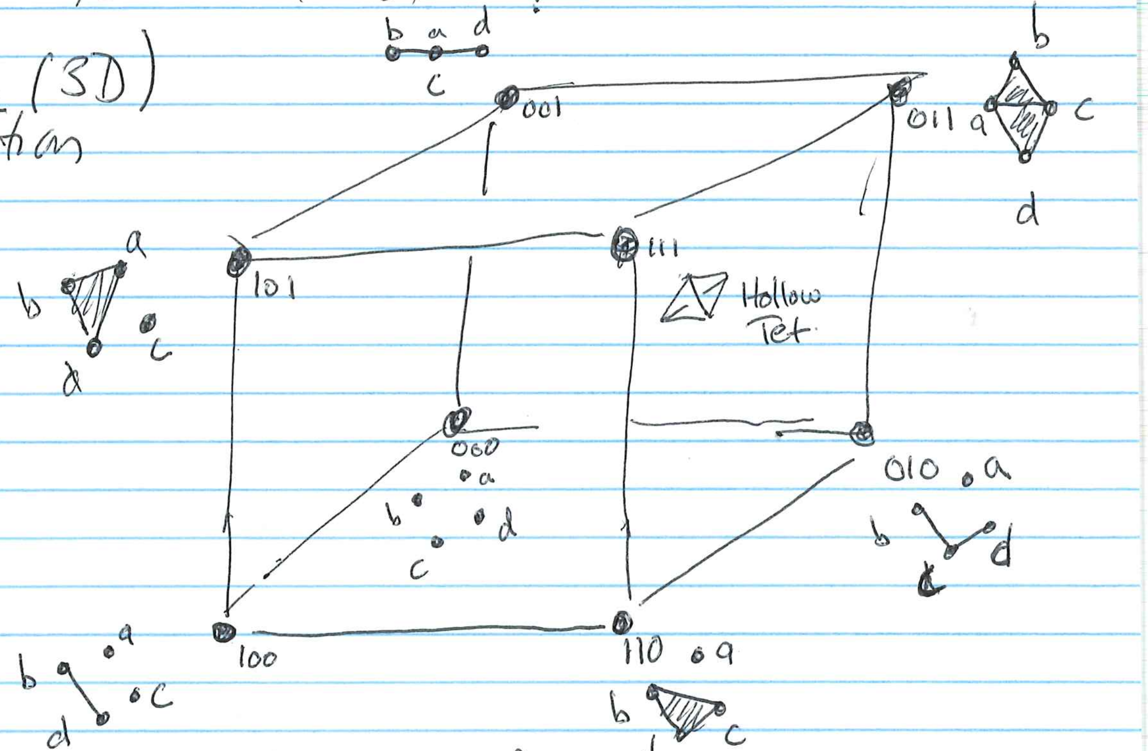
§0. <u>Question</u> How do the homology modules H_i relate to each other?

<u>Example (3D)</u>
<u>filtration</u>

$R = k[x, y, z]$



$$0 \to R^4 \xrightarrow{\partial_2} R^6$$

		011 abc	101 abd	011 acd	110 bcd
001	ab	y	x	0	0
011	ac	-1	0	1	0
001	ad	0	$-x$	$-y$	0
010	bc	z	0	0	x
100	bd	0	z	0	$-y$
010	cd	0	0	z	z

$$R^6 \xrightarrow{\partial_1} R^4 \to 0$$

	001 ab	011 ac	001 ad	010 bc	100 bd	010 cd
a	$-z$	$-yz$	$-z$	0	0	0
b	z	0	0	$-y$	$-x$	0
c	0	yz	0	y	0	$-y$
d	0	0	z	0	x	y

<u>Example</u>: $[abc]$ is in degree $\boxed{011}$

$$\partial_2(\underbrace{[abc]}_{011}) = \underbrace{\quad}_{[010]} \underbrace{[ab]}_{001} \underbrace{\quad}_{1} - \underbrace{[ac]}_{011} \underbrace{\quad}_{[001]} \underbrace{[bc]}_{010}$$

$$= y[ab] - [ac] + z[bc] = \underline{\underline{\text{column } 1}}$$

<u>Check</u> $\begin{pmatrix} x \\ -y \\ x \\ -z \end{pmatrix} \in \ker \partial_2 \Rightarrow H_2 \simeq \boxed{R \text{ in degree } (1,1,1)}$

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It turns out there are relations among the H_i. But they are pretty complex, and lead us into the realm of derived functors and spectral sequences.

§1 Derived Functors (as usual, turbo version ☺)

Def First, recall that a functor between categories

$$\mathcal{A} \xrightarrow{\mathcal{F}} \mathcal{B}$$

takes $A_1 \xrightarrow{f} A_2 \longmapsto \mathcal{F}(A_1) \xrightarrow{F(f)} \mathcal{F}(A_2)$

$ob(\mathcal{A}) \longrightarrow ob(\mathcal{B})$
$morph(\mathcal{A}) \to morph(\mathcal{B})$ is covariant if arrow → preserved; contravariant → reversed

and Left (right) exact if a s.e.s in $\mathcal{A}$

$$0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0 \overset{\text{L. exact, covariant}}{\Longrightarrow} 0 \to \mathcal{F}(A_1) \xrightarrow{F(f_1)} \mathcal{F}(A_2) \xrightarrow{F(f_2)} \mathcal{F}(A_3) \; \triangle$$

? **Question** ? What goes here

Answer: Derived functors

4 flavors: for co/contra L/R exact.

F	covar.	contra v.
L exact	I	P
R exact	P	I

(R noeth. comm ring w/unit)

Henceforth, we work in the cat. of R-mod

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Def: An object P is *projective* if $\forall$ $\begin{tikzcd} & P \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$ we have

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed, "\exists \phi"'] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

example: free module

An object I is *injective* if $\forall$ $\begin{tikzcd} A & B \arrow[l] \arrow[d] & 0 \arrow[l] \\ & I & \end{tikzcd}$ we have

$$\begin{tikzcd} A \arrow[dr, dashed, "\exists \phi"'] & B \arrow[l] \arrow[d] & 0 \arrow[l] \\ & I & \end{tikzcd}$$

Def/Recipe: If F is a L. exact, contravariant functor (e.g. $\mathrm{Hom}_R(\cdot, R)$) then we define $R^iF(M)$ for an object (R-mod) M via:

1. Take a projective resolution (exact sequence) $0 \leftarrow M - P_0 \leftarrow P_1 \leftarrow \cdots$, for M
2. Drop M, apply F.
3. $R^iF(M) = H_i(0 \to F(P_0) \to F(P_1) \to F(P_2) \to \cdots)$

Seems convoluted and complicated:

Example $F = \mathrm{Hom}_R(\cdot, R)$ is L. exact, contravariant.

$R^i(\mathrm{Hom}(M, R))$ is *called* $\mathrm{Ext}^i(M, R)$

Suppose $M = k[x,y]/(x^2, xy)$, $R = k[x,y]$.

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<u>Recipe:</u> Free resolutn of $\overset{R}{k[x,y]}/x^2,xy = M$

$$0 \leftarrow M \leftarrow R \xleftarrow{[x^2,xy]} R^2(-2) \xleftarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} R(-3) \leftarrow 0$$

Drop M, $\mathrm{Hom}_R(\bullet, R)$

$$\Rightarrow H_i\left(0 \to R \xrightarrow{\begin{bmatrix} x^2 \\ xy \end{bmatrix}} R^2(2) \xrightarrow{[y,-x]} R(3) \to 0 \right)$$

$H_0 = 0 = \mathrm{Ext}^0$

$H_1 = \ker(y,-x) / \mathrm{im}\begin{pmatrix} x^2 \\ xy \end{pmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} / x\begin{bmatrix} x \\ y \end{bmatrix} \simeq R/x.$

$H_2 = R(3) / \mathrm{im}\,[y,-x] \simeq k(3) = R(3)/(x,y)$

//

<u>Aside:</u> A beautiful theorem of Eisenbud-Huneke-Vasconcelos says $P \in \mathrm{Ass}\, M$ of codim $i \Leftrightarrow p \in \mathrm{Ass}(\mathrm{Ext}^i(M,R))$

we see the ass. primes of M $\{\langle x \rangle, \langle x,y \rangle\}$ appear over Ext^1 and Ext^2.

$$\boxed{\text{So What?}}$$

<u>Exercise:</u> $R^iF(M)$
- Does not depend on choice $P_\bullet$
- $0 \to A \to B \to C \to 0$ gives a l.e.s $\cdots \to R^{i-1}F(A) \to R^iF(C) \to R^i(B) \to R^iF(A) \to R^{i+1}F(C) \to$
- is zero if M is projective.

(say F is L. exact, contrav)

(sim versions for others)

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Deep Breath: Our motivating question is "How are the $H_i(MPH)$ related". The answer uses Ext, and also a *technical* tool:

§2 Spectral Sequences:

Suppose we have a

Def Double complex:

$$F_{\bullet\bullet} = \begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ F_{02} \arrow[d] & F_{12} \arrow[l] \arrow[d] & F_{22} \arrow[l] \arrow[d] & \cdots \\ F_{01} \arrow[d] & F_{11} \arrow[l] \arrow[d] & F_{21} \arrow[l] \arrow[d] & \cdots \\ F_{00} & F_{10} \arrow[l] & F_{20} \arrow[l] & \cdots \end{tikzcd}$$

$d_v = d_{vert}$, $d_h = d_{hor}$.

How do vertical and horizontal Homologies relate?

① F_{tot} is a single complex, $F_m = \bigoplus_{i+j=m} F_{ij}$ (sum along diagonals)

$D_m = d_h + (-1)^m d_v$. Exercise $D^2 = 0$

② A filtration of a module M is $0 \subseteq M_0 \subseteq M_1 \subseteq \ldots \subseteq M$ and we define $gr(M)$ "associated graded" as $\bigoplus M_i/M_{i+1}$

③ $\boxed{\text{Big Picture}}$ We'll compare two filtrations of $H_k(F_{tot})$

Def ④ ${}_{hor}E^1_{ij}$ = Homology at position (i,j) wrt d_{hor}

${}_{vert}E^1_{ij}$ = " wrt d_{vert}

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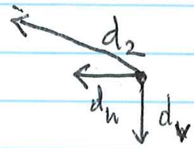
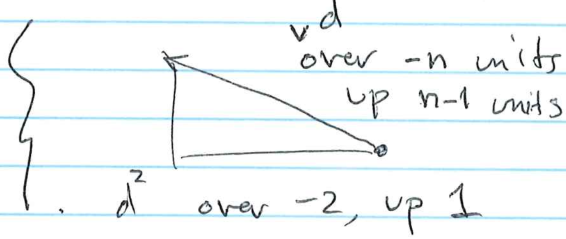
If we look at $\leftarrow {}_vE^1_{ij} \leftarrow$ there is still a horizontal diff.

defn ${}_vE^2_{ij}$ = Homology (wrt d_{hor}) of this

Seems like we've burned all our diffs. But there is a diff 

coming from snake lemma "Knights move"

Hard ex: Snake lemma propogates

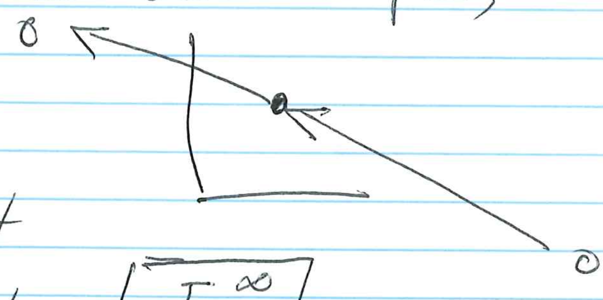


E^n_{ij} homology at n^{th} step

$\downarrow {}_vd^n_{ij}$

$E^n_{i-n,\, j+n-1}$

For a 1^{st} quadrant double complex, eventually



nothing in/out so it stabilizes $\boxed{{}_vE^\infty_{ij}}$

Defn: If $gr(M)_m = \bigoplus_{i+j=m} E^\infty_{ij}$, we say the spectral sequence converges $E^r \Rightarrow M$

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This is <u>symmetric</u>.

<u>Punchline:</u>

$$\bigoplus {}_h E^\infty_{ij} \quad \text{and} \quad \bigoplus {}_v E^\infty_{ij}$$

are associated gradeds of same object (if ss converges) $\boxed{\text{BUT}}$ <u>individual</u> <u>summands</u> differ.

Get mileage from s.s. by playing off the two filtrations

$\boxed{\text{Example}}$ Les in homology via spectral seq.

$$\left\{ \begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 & A_0 \arrow[l] \arrow[d] & A_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & B_0 \arrow[l] \arrow[d] & B_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & C_0 \arrow[l] \arrow[d] & C_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 &
\end{tikzcd} \right. \quad \text{s.e.s of complexes.}$$

<u>Columns exact</u> $\Rightarrow$ ${}_v E^1_{ij} = 0 = {}_v E^\infty_{ij}$ ☹

${}_h E^1_{ij}$:

$$\begin{tikzcd}
H_0(A) \arrow[d, "\alpha_0"'] & H_1(A) \arrow[d, "\alpha_1"] & \\
H_0(B) \arrow[d, "\beta_0"'] & H_1(B) \arrow[d, "\beta_1"] & \cdots \\
H_1(B) & H_1(C) &
\end{tikzcd}$$

${}_h E^2_{ij}$

$$\begin{tikzcd}
0 \arrow[dr, "d_2"'] & & \\
 & \ker \alpha_0 \arrow[ddr] & \ker \alpha_1 \\
 & \ker \beta_0 / \operatorname{im} \alpha_0 \arrow[ddr] & \ker \beta_1 / \operatorname{im} \alpha_1 \\
 & \operatorname{cok} \beta_0 & \operatorname{cok} \beta_1 \\
 & & 0
\end{tikzcd}$$

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Key:

$$\begin{array}{l} 0 \xrightarrow{d_2} \ker\beta_1/\operatorname{im}\alpha_0 \xrightarrow{d_2} 0 \end{array} \Rightarrow {}_hE^2_{0,1} = {}_hE^\infty_{01} \subseteq \bigoplus_{i+j=1} E^\infty_{ij} = 0$$

$\Rightarrow \ker\beta_1 = \operatorname{im}\alpha_0$

$\Rightarrow H_0(A) \to \underbrace{H_0 B}_{\boxed{\text{exact here}}} \to H_0(C)$

What about $\ker(\alpha_0) \xrightarrow{d_2} \operatorname{coker}\beta_1$?

$E^3_{02} = \ker d_2$

$E^3_{10} = \operatorname{coker} d_2$

but these are summands of somethig equal to 0.

$\Rightarrow \ker d_2 = 0 = \operatorname{coker} d_2$

$\Rightarrow$ d_2 is an isomorphism, so we can patch together a map.

$\Rightarrow H_1(C) \to H_0(A)$ $\boxed{\text{is exact}}$.

$\ker(H_0(A) \to H_0(B)) \overset{d_2}{/\!/} \cancel{A} = \operatorname{coker}(H_1 B \to H_1 C)$

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THM (Appendix of these notes for a proof)

Grothendieck SS of composite functors

$C_1 \xrightarrow{F} C_2 \xrightarrow{G} C_3$ Left exact, covariant

If $A \in C_1$ has an F-acyclic resolution A^i such that $F(A^i)$ is G-acyclic, then

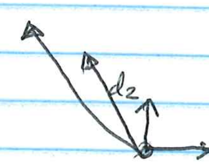
$$R^iG(R^jF(A)) \Rightarrow R^{i+j}(GF(A))$$

For our example, the functors are Homology and Ext,

$$\begin{tikzcd}
\mathrm{Ext}^3(H_0,R) & \mathrm{Ext}^3(H_1,R) & \mathrm{Ext}^3(H_2,R) \\
\mathrm{Ext}^2(H_0,R) & \mathrm{Ext}^2(H_1,R) & \mathrm{Ext}^2(H_2,R) \\
\mathrm{Ext}^1(H_0,R) & \mathrm{Ext}^1(H_1,R) & \mathrm{Ext}^1(H_2,R) \\
\mathrm{Ext}^0(H_0(R) & \mathrm{Ext}^0(H_1,R) & \mathrm{Ext}^0(H_2,R) \arrow[uul, "d_2"'] \arrow[uuull, "d_3"]
\end{tikzcd}$$

all zero

can show all zero



Get a nontrivial map from $\mathrm{Hom}(H_2,R)$ to $\mathrm{Ext}^3(H_0,R)$!

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Appendix:

Thm (Grothendieck s.s.) If $A \in C$, has an F-acyclic resoln A^i such that $F(A^i)$ is G-acyclic, then

$$R^iG(R^jF(A)) \Rightarrow R^{i+j}(GF(A))$$

//

Def: an injective resoln of a complex

$$0 \to A^0 \to A^1 \to A^2 \to \cdots$$

is

$$\begin{array}{ccccc} I^{02} & \to & I^{12} & & \\ \uparrow & & \uparrow & & \\ I^{01} & \to & I^{11} & & \cdots \\ \uparrow & & \uparrow & & \\ I^{00} & \to & I^{10} & & \\ \hline A_0 & & A_1 & & A_2 \end{array}$$

- a double complex (commutes)
- s.t. cols are inj res of A^i
- $\ker d_h^{jk}$ is an injective summand of I^{jk}.

Since $\ker d^{jk}$ is an injective summand of I^{jk}, $\mathrm{im}\, d^{jk}$ is also injective, so

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{im}\, d^{j-1,k} \arrow[r] \arrow[d] & \ker d^{jk} \arrow[r] \arrow[dl] & H_d^{jk} \arrow[r] & 0 \\ & \mathrm{im}\, d^{j-1,k} & & & \end{tikzcd}$$

So $I^{jk} = \ker d^{jk} \oplus \mathrm{im}\, d^{jk}$

$$\ker d^{jk} = H_d^{jk} \oplus \mathrm{im}\, d^{j-1,k}.$$

Flipping as we go

$$\begin{array}{ccccc}
I^{j-1,k} & \longrightarrow & I^{j,k} & \longrightarrow & I^{j+1,k} \\
\| & & \| & & \| \\
\operatorname{Im} d^{j-2,k} & \xrightarrow{0} & \operatorname{Im} d^{j,k} & \xrightarrow{1} & \operatorname{Im} d^{jk} \\
\oplus & & \oplus & & \oplus \\
H^{j-1,k} & \xrightarrow{0} & H^{j,k} & \xrightarrow{0} & H^{j-1,k} \\
\oplus & & \oplus & & \oplus \\
\operatorname{Im} d^{j-1,k} & \xrightarrow{1} & \operatorname{Im} d^{j-1,k} & \xrightarrow{0} & \operatorname{Im} d^{j+1,k}
\end{array} \Bigg\} \star$$

Keep on board

By "Horseshoe lemma", can do so the H^{jk} are an inj res of $H(A^\circ)$, called a Cartan–Eilenberg resoln.

Pf (GSS) Take res of A by F-acyclics, hit it with F, take inj res [Cartan Eilenberg] of result.

$$\begin{array}{ccccc}
\vdots & & & & \\
I^{01} & \to & I^{11} & & \cdots \\
\uparrow & & \uparrow & & \\
I^{00} & \to & I^{10} & & \\
\uparrow & & \uparrow & & \\
F(A^0) & \to & F(A^1) & \to & \cdots
\end{array}$$

Using above $\star$ the I^{jk} decompose

$${}_hE^1_{ij} = G(H^{ij})$$

j^{th} object in inj res of i^{th} coho of $F(A^\cdot)$, so

$${}_hE^2_{ij} = H^j\left(0 \to G(H^{i0}) \to G(H^{i,1}) \to \cdots\right)$$
$$= R^jG(R^iF(A))$$

How about vertical filtratn?

Appendix

Each $I^{i\cdot}$ is an inj res of $F(A^i)$,

so ${}_{\mathrm{I}}E^{1}_{ij} = H^j(0 \to G(I^{i0}) \to G(I^{i1}) \to \cdots)$

We assumed the $F(A^i)$ are G-acyclic,

so $R^jG(F(A^i)) = 0$, $j > 0$

$$\Rightarrow {}_{\mathrm{I}}E^{2}_{ij} = \begin{cases} H(GF(A^i)) & j=0 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow {}_{\mathrm{I}}E^{\infty}_{ij} = \begin{cases} R^{i+j}(GF(A)) & j=0 \\ 0 & \text{else} \end{cases}$$

∎