The Mathematics of Topological Data Analysis
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Preface

This book is a mirror of data science: it covers a wide range of topics, at levels of sophistication varying from the elementary (matrix algebra) to the esoteric (Grothendieck spectral sequence). My hope is that there is something for everyone, from undergraduates immersed in a first linear algebra class to sophisticates investigating zig-zag persistent homology. Readers are encouraged to barhop; the goal is to give an intuitive and hands on introduction to the topics, rather than a punctiliously precise presentation.

The notes grew out of a class on Topological Data Analysis taught to statistics graduate students at Auburn University during the COVID summer of 2020. The book reflects that: it is written for a mathematically engaged audience interested in understanding some of the algebraic and topological aspects of data analysis. Because data analysis draws practitioners with a broad range of experience, the book assumes little background at the outset. However, the advanced topics in the latter part of the book require a willingness to tackle technically difficult material.

There are a plethora of excellent texts on basic algebra and topology which the reader may want to use to supplement these notes, some are listed below. Theorems and proofs which are sketched or omitted in these notes may be found there.

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Techniques from linear algebra have been essential tools in data analysis from the
birth of the field, and so the book kicks off with the basics:

- Least Squares Approximation
- Covariance Matrix and Spread of Data
- Singular Value Decomposition

Tools from topology have recently made an impact on data analysis. This text
provides the background to understand developments such as persistent homology.
Suppose we are given point cloud data, that is, a set of points $X$:

![Point Cloud Data]

If $X$ was sampled from some object $Y$, we’d like to use $X$ to infer properties
of $Y$. Persistent Homology applies tools of algebraic topology to do this. We start
by using $X$ as a seed from which to build a family of spaces

$$X_\epsilon = \bigcup_{p \in X} N_\epsilon(p),$$

where $N_\epsilon(p)$ denotes an $\epsilon$ ball around $p$.

As $X_\epsilon \subseteq X_{\epsilon'}$ if $\epsilon \leq \epsilon'$, we have a family of topological spaces and inclusion maps.

![Family of Spaces]

As Weinberger notes in [98], persistent homology is a type of Morse theory:
there are a finite number of values of $\epsilon$ where the topology of $X_\epsilon$ changes. Notice that when $\epsilon \gg 0$, $X_\epsilon$ is a giant blob; so $\epsilon$ is typically restricted to a range $[0, x]$. Topological features which “survive” to the parameter value $x$ are said to be persistent; in the example above the circle $S^1$ is a persistent feature.

The first three chapters of this book are an algebra boot camp. Chapter 1 is
pure linear algebra, and provides a brisk review of the tools from linear algebra
most applicable to data analysis. Chapter 2 and 3 cover, essentially, the content of
a one-semester upper level undergraduate class in algebra. Chapter 3 is devoted
to modules over a principal ideal domain; this is because the structure theorem for
modules over a principal ideal domain plays a central role in persistent homology.
Chapter 4 begins with the definition of a topological space, and—in a dizzying leap in level of sophistication—closes with a description of how to visualize a sheaf on an algebraic variety. Algebraic topology enters in Chapter 5, and is complemented by the introduction of key tools from homological algebra in Chapter 6. Chapters 7 and 8 cover, respectively, persistent and multiparameter persistent homology. Chapter 9 is the pièce de résistance (or perhaps the coup de grâce)—a quick and dirty guide to derived functors and spectral sequences.

There are a number of texts which tackle data analysis from the perspective of pure mathematics. *Elementary Applied Topology* by Rob Ghrist is closest in spirit to these notes; it has a more topological flavor (as well as wonderfully illuminating illustrations!); another excellent text is *Computational Topology* by Herbert Edelsbrunner and John Harer. At the other end of the spectrum, *Persistence theory: from quiver representations to data analysis* by Steve Oudot is intended for more advanced readers. Two central aspects of data analysis—statistics and computational methods—are not treated in these notes; each deserves a separate volume.

Many folks deserve acknowledgement for their contributions to this book, starting with my collaborators in the area: Heather Harrington, Nina Otter, and Ulrike Tillmann. A precursor to the summer TDA class was a minicourse I taught at CIMAT, organized by Abraham Martín del Campo, Antonio Rieser, and Carlos Vargas Obieta; the summer TDA class was itself made possible by the Rosemary Kopel Brown endowment. Members of the TDA community and applied algebraic topology research network (AATRN) have provided lots of useful feedback, special thanks to FILL IN, and Ketai Chen.

Data science is a moving target: one of today’s cutting edge tools may be relegated to the ash bin of history tomorrow. For this reason the text aims to highlight mathematically important concepts which have proven or potential utility in data analysis. But mathematics is a human endeavor, so it is wise to remember Seneca: “Omnia humana brevia et caduca sunt”.

*Hal Schenck*
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Chapter 1

Linear Algebra Tools for Data Analysis

We begin with a short and intense review of linear algebra. There are a number of reasons for this approach; first and foremost is that linear algebra is the computational engine that drives most of mathematics, from numerical analysis (finite element method) to algebraic geometry (Hodge decomposition) to statistics (covariance matrix and the shape of data). A second reason is that practitioners of data science come from a wide variety of backgrounds—a statistician may not have seen eigenvalues since their undergraduate days. The goal of this chapter is to develop basic dexterity dealing with

- Linear Equations, Gaussian Elimination, Matrix Algebra.
- Vector Spaces, Linear Transformations, Basis and Change of Basis.
- Diagonalization, Webpage Ranking, Data and Covariance.
- Orthogonality, Least Squares Data Fitting, Singular Value Decomposition.

There are entire books written on linear algebra, so our focus will be on examples and computation, with proofs either sketched or left as exercises.

1. Linear Equations, Gaussian Elimination, Matrix Algebra

In this section, our goal is to figure out how to set up a system of equations to study the following question:

**Example 1.1.** Suppose each year in Smallville that 30% of nonsmokers begin to smoke, while 20% of smokers quit. If there are 8000 smokers and 2000 nonsmokers at time $t = 0$, after 100 years, what are the numbers? After $n$ years? Is there an equilibrium state at which the numbers stabilize?
It turns out that linear algebra provides a terrific toolkit to analyze this question. We begin with the basics of solving a system of linear equations such as
\[
\begin{align*}
  x - 2y &= 2 \\
  2x + 3y &= 6
\end{align*}
\]

*Gaussian elimination* provides a systematic way to manipulate the equations to reduce to fewer equations in fewer variables. A linear equation (no variable appears to a power greater than one) in one variable is of the form \( x = \text{constant} \), so is trivial. For the system above, we can eliminate the variable \( x \) by multiplying the first row by \(-2\) (note that this does not change the solutions to the system), and adding it to the second row, yielding the new equation \( 7y = 2 \). Hence \( y = \frac{2}{7} \), and substituting this for \( y \) in either of the two original equations, we solve for \( x \), finding that \( x = \frac{18}{7} \).

Gaussian elimination is a formalization of this simple example: given a system of linear equations
\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  \vdots & \quad \vdots \quad \vdots
\end{align*}
\]

(a) swap order of equations so \( a_{11} \neq 0 \).

(b) multiply the first row by \( \frac{1}{a_{11}} \), so the coefficient of \( x_1 \) in the first row is 1.

(c) subtract \( a_{i1} \) times the first row from row \( i \), for all \( i \geq 2 \).

At the end of this process, only the first row has an equation with nonzero \( x_1 \) coefficient. Hence, we have reduced to solving a system of fewer equations in fewer unknowns. Iterate the process. It is important that the operations above do not change the solutions to the system of equations; they are known as *elementary row operations*: formally, these operations are

(a) Interchange two rows.

(b) Multiply a row by a constant.

(c) Add a multiple of one row to another row.

**Exercise 1.2.** Solve the system
\[
\begin{align*}
  x + y + z &= 3 \\
  2x + y &= 7 \\
  3x + 2z &= 5
\end{align*}
\]

You should end up with \( z = -\frac{7}{5} \), now backsolve. It is worth thinking about the geometry of the solution set. Each of the three equations defines a plane in \( \mathbb{R}^3 \). What are the possible solutions to the system? If two planes are parallel, we have equations \( ax + by + cz = d \) and \( ax + by + cz = e \), with \( d \neq e \), then there are no solutions, since no point can lie on both planes. On the other hand, the three planes could meet in a single point—this occurs for the system \( x = y = z = 0 \), for which the origin \( (0, 0, 0) \) is the only solution. Other geometric possibilities for the set of
There is a simple shorthand for writing a system of linear equations as above using matrix notation. To do so, we need to define matrix multiplication.

**Definition 1.3.** A matrix is a $m \times n$ array of elements, where $m$ is the number of rows and $n$ is the number of columns.

We give a formal definition of a vector shortly; informally we think of a real vector $v$ as an $n \times 1$ matrix with entries in $\mathbb{R}$ and visualize it as a directed arrow with tail at the origin $(0, \ldots, 0)$ and head at the point of $\mathbb{R}^n$ corresponding to $v$.

**Definition 1.4.** The dot product of vectors $v = [a_1, \ldots, a_n]$ and $w = [b_1, \ldots, b_n]$ is

$$v \cdot w = \sum_{i=1}^{n} a_i b_i,$$

and the length of $v$ is $|v| = \sqrt{v \cdot v}$.

By the law of cosines, $v, w$ are orthogonal iff $v \cdot w = 0$, which you’ll prove in Exercise 4.1. An $m \times n$ matrix $A$ and $p \times q$ matrix $B$ can be multiplied when $n = p$. If $(AB)_{ij}$ denotes the $(i, j)$ entry in the product matrix, then

$$(AB)_{ij} = \text{row}_i(A) \cdot \text{col}_j(B).$$

This definition may seem opaque. It is set up exactly so that when the matrix $B$ represents a transition from State$_1$ to State$_2$ and the matrix $A$ represents a transition from State$_2$ to State$_3$, then the matrix $AB$ represents the composite transition from State$_1$ to State$_3$. This makes clear the reason that the number of columns of $A$ must be equal to the number of rows of $B$ to compose the operations: the target of the map $B$ is the source of the map $A$.

**Exercise 1.5.**

$$\begin{bmatrix} 2 & 7 \\ 3 & 3 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 37 & 46 & 55 & 64 \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Fill in the remainder of the entries.

**Definition 1.6.** The transpose $A^T$ of a matrix $A$ is defined via $(A^T)_{ij} = A_{ji}$. $A$ is symmetric if $A^T = A$, and diagonal if $A_{ij} \neq 0 \Rightarrow i = j$. If $A$ and $B$ are diagonal $n \times n$ matrices, then

$$AB = BA \text{ and } (AB)_{ii} = a_{ii} \cdot b_{ii}.$$ 

The $n \times n$ identity matrix $I_n$ is a diagonal matrix with 1’s on the diagonal; if $A$ is an $n \times m$ matrix, then

$$I_n \cdot A = A = A \cdot I_m.$$ 

An $n \times n$ matrix $A$ is invertible if there exists an $n \times n$ matrix $B$ such that $BA = AB = I_n$. We write $A^{-1}$ for the matrix $B$; the matrix $A^{-1}$ is the inverse of $A$. 

common solutions are a line or plane; describe the algebra that corresponds to the last two possibilities. ∙
Exercise 1.7. Find a pair of $2 \times 2$ matrices $(A, B)$ such that $AB \neq BA$. Show that $(AB)^T = B^T A^T$, and use this to prove that the matrix $A^T A$ is symmetric. ♦

In matrix notation a system of $n$ linear equations in $m$ unknowns is written as

$$A \cdot x = b,$$

where $A$ is an $n \times m$ matrix and $x = [x_1, \ldots, x_m]^T$.

We close this section by returning to our vignette

Example 1.8. To start the analysis of smoking in Smallville, we write out the matrix equation representing the change during the first year, from $t = 0$ to $t = 1$.

Let $[n(t), s(t)]^T$ be the vector representing the number of nonsmokers and smokers (respectively) at time $t$. Since 70% of nonsmokers continue as nonsmokers, and 20% of smokers quit, we have $n(1) = .7 n(0) + .2 s(0)$. At the same time, 30% of nonsmokers begin smoking, while 80% of smokers continue smoking, hence $s(1) = .3 n(0) + .8 s(0)$. We encode this compactly as the matrix equation:

$$
\begin{bmatrix}
  n(1) \\
  s(1)
\end{bmatrix} =
\begin{bmatrix}
  .7 & .2 \\
  .3 & .8
\end{bmatrix}
\begin{bmatrix}
  n(0) \\
  s(0)
\end{bmatrix}
$$

Now note that to compute the smoking status at $t = 2$, we have

$$
\begin{bmatrix}
  n(2) \\
  s(2)
\end{bmatrix} =
\begin{bmatrix}
  .7 & .2 \\
  .3 & .8
\end{bmatrix}
\begin{bmatrix}
  n(1) \\
  s(1)
\end{bmatrix} =
\begin{bmatrix}
  .7 & .2 \\
  .3 & .8
\end{bmatrix}
\begin{bmatrix}
  .7 & .2 \\
  .3 & .8
\end{bmatrix}
\begin{bmatrix}
  n(0) \\
  s(0)
\end{bmatrix}
$$

And so on, ad infinitum. Hence, to understand the behavior of the system for $t$ very large (written $t \gg 0$), we need to compute

$$
\left( \lim_{t \to \infty} \begin{bmatrix}
  .7 & .2 \\
  .3 & .8
\end{bmatrix} \right)^t \begin{bmatrix}
  n(0) \\
  s(0)
\end{bmatrix}
$$

Matrix multiplication is computationally expensive, so we’d like to find a trick to save ourselves time, energy, and effort. The solution is the following, which for now will be a Deus ex Machina (but lovely nonetheless!) Let $A$ denote the $2 \times 2$ matrix above (multiply by 10 for simplicity). Then

$$
\begin{bmatrix}
  3/5 & -2/5 \\
  1/5 & 1/5
\end{bmatrix}
\begin{bmatrix}
  7 & 2 \\
  3 & 8
\end{bmatrix}
\begin{bmatrix}
  1 & 2 \\
 -1 & 3
\end{bmatrix} =
\begin{bmatrix}
  5 & 0 \\
  0 & 10
\end{bmatrix}
$$

Write this equation as $BAB^{-1} = D$, with $D$ denoting the diagonal matrix on the right hand side of the equation. An easy check shows $BB^{-1}$ is the identity matrix $I_2$. Hence

$$(BAB^{-1})^n = (BAB^{-1})(BAB^{-1})(BAB^{-1}) \cdots BAB^{-1} = BA^n B^{-1} = D^n,$$
which follows from collapsing the consecutive $B^{-1}B$ terms in the expression. So

$$BA^nB^{-1} = D^n,$$

and therefore $A^n = B^{-1}D^nB$.

As we saw earlier, multiplying a diagonal matrix times itself costs nothing, so we have reduced a seemingly costly computation to almost nothing; how do we find the mystery matrix $B$? The next two sections of this chapter are devoted to answering this question.

2. Vector Spaces, Linear Transformations, Basis and Change of Basis

In this section, we lay out the formal underpinnings of linear algebra, beginning with the definition of a vector space over a field $\mathbb{K}$. A field is a type of ring, and is defined in detail in the next chapter. For our purposes, the field $\mathbb{K}$ will typically be one of $\{\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/p\}$, where $p$ is a prime number.

**Definition 2.1.** A vector space $V$ is a collection of objects (vectors), endowed with two operations: vectors can be added to produce another vector, or multiplied by an element of the field. Hence the set of vectors $V$ is closed under the operations.

**Example 2.2.** Examples of vector spaces.

(a) $V = \mathbb{K}^n$, with $[a_1, \ldots, a_n] + [b_1, \ldots, b_n] = [a_1 + b_1, \ldots, a_n + b_n]$ and $c[a_1, \ldots, a_n] = [ca_1, \ldots, ca_n]$.

(b) The set of polynomials of degree at most $n - 1$, with coefficients in $\mathbb{K}$. Show this has the same structure as part (a).

(c) The set of continuous functions on the unit interval.

Harking back to our earlier description of a vector as a directed arrow, we can visualize vector addition as putting the tail of one vector at the head of another: draw a picture to convince yourself that

$$[1, 2] + [2, 4] = [3, 6].$$

2.1. Basis of a Vector Space.

**Definition 2.3.** A set of vectors $\{v_1, \ldots, v_k\} \subseteq V$ a vector space is

- linearly independent (or independent) if
  \[
  \sum_{i=1}^{k} a_i v_i = 0 \Rightarrow \text{all the } a_i = 0.
  \]

- a spanning set for $V$ (or spans) if for any $v \in V$ there exist $a_i \in \mathbb{K}$ such that
  \[
  \sum_{i=1}^{k} a_i v_i = v.
  \]
Example 2.4. The set \{[1, 0], [0, 1], [2, 3]\} is dependent, since $2 \cdot [1, 0] + 3 \cdot [0, 1] - 1 \cdot [2, 3] = [0, 0]$. It is a spanning set, since an arbitrary vector \([a, b] = a \cdot [1, 0] + b \cdot [0, 1]\). On the other hand, for $V = \mathbb{K}^3$, the set of vectors \{[1, 0, 0], [0, 1, 0]\} is independent, but does not span.

Definition 2.5. A subset $S = \{v_1, \ldots, v_k\} \subseteq V$ is a basis for $V$ if it spans and is independent. If $S$ is finite, we define the dimension of $V$ to be the cardinality of $S$.

A basis is loosely analogous to a set of letters for a language where the words are vectors. The spanning condition says we have enough letters to write every word, while the independent condition says the representation of a word in terms of the set of letters is unique. The vector spaces we encounter in this book will be finite dimensional. A cautionary word: there are subtle points which arise when dealing with infinite dimensional vector spaces.

Exercise 2.6. Show that the dimension of a vector space is well defined.

2.2. Linear Transformations. One of the most important constructions in mathematics is that of a mapping between objects. Typically, one wants the objects to be of the same type, and for the map to preserve their structure. In the case of vector spaces, the right concept is that of a linear transformation:

Definition 2.7. Let $V$ and $W$ be vector spaces. A map $T : V \rightarrow W$ is a linear transformation if

$$T(cv_1 + v_2) = c T(v_1) + T(v_2),$$

for all $v_1 \in V, c \in \mathbb{K}$. Put more tersely, sums split up, and scalars pull out.

While a linear transformation may seem like an abstract concept, the notion of basis will let us represent a linear transformation via matrix multiplication. On the other hand, our vignette about smokers in Smallville in the previous section shows that not all representations are equal. This brings us to change of basis.

Example 2.8. The sets $B_1 = \{[1, 0], [1, 1]\}$ and $B_2 = \{[1, 1], [1, -1]\}$ are easily checked to be bases for $\mathbb{K}^2$. Write $v_{B_i}$ for the representation of a vector $v$ in terms of the basis $B_i$. For example

$$[0, 1]_{B_1} = 0 \cdot [1, 0] + 1 \cdot [1, 1] = 1 \cdot [1, 1] + 0 \cdot [1, -1] = [1, 0]_{B_2}$$

The algorithm to write a vector $b$ in terms of a basis $B = \{v_1, \ldots, v_n\}$ is as follows: construct a matrix $A$ whose columns are the vectors $v_i$, then use Gaussian elimination to solve the system $Ax = b$.

Exercise 2.9. Write the vector $[2, 1]$ in terms of the bases $B_1$ and $B_2$ above. ⬤
To represent a linear transformation \( T : V \to W \), we need to have frames of reference for the source and target—this means choosing bases \( B_1 = \{ v_1, \ldots, v_n \} \) for \( V \) and \( B_2 = \{ w_1, \ldots, w_m \} \) for \( W \). Then the matrix \( M_{B_2B_1} \) representing \( T \) with respect to input in basis \( B_1 \) and output in basis \( B_2 \) has as \( i \)th column the vector \( T(v_i) \), written with respect to the basis \( B_2 \). An example is in order:

**Example 2.10.** Let \( V = \mathbb{R}^2 \), and let \( T \) be the transformation that rotates a vector counterclockwise by 90 degrees. With respect to the standard basis \( B_1 = \{ [1, 0], [0, 1] \} \), \( T([1, 0]) = [0, 1] \) and \( T([0, 1]) = [-1, 0] \), so

\[
T_{B_1B_1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

Using the bases \( B_1 \) (for input) and \( B_2 \) (for output) from Example 2.8 yields

\[
T([1, 0]) = [0, 1] = 1/2 \cdot [1, 1] - 1/2 \cdot [-1, -1] \\
T([1, 1]) = [-1, 1] = 0 \cdot [1, 1] - 1 \cdot [-1, -1]
\]

So

\[
M_{B_2B_1} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & -1 \end{bmatrix}
\]

### 2.3. Change of Basis

Suppose we have a representation of a matrix or vector with respect to basis \( B_1 \), but need the representation with respect to basis \( B_2 \). This is analogous to a German speaking diner being presented with a menu in French: we need a translator (or language lessons!)

**Definition 2.11.** Let \( B_1 \) and \( B_2 \) be two bases for the vector space \( V \). The change of basis matrix \( \Delta_{21} \) takes as input a vector represented in basis \( B_1 \), and outputs the same vector represented with respect to the basis \( B_2 \).

**Algorithm 2.12.** Given bases \( B_1 = \{ v_1, \ldots, v_n \} \) and \( B_2 = \{ w_1, \ldots, w_n \} \) for \( V \), to find the change of basis matrix \( \Delta_{21} \), form the \( n \times 2n \) matrix whose first \( n \) columns are \( B_2 \) (the “new” basis), and whose second \( n \) columns are \( B_1 \) (the “old” basis). Row reduce to get a matrix whose leftmost \( n \times n \) block is the identity. The rightmost \( n \times n \) block is \( \Delta_{21} \). The proof below for \( n = 2 \) generalizes easily.

**Proof.** Since \( B_2 \) is a basis, we can write

\[
v_1 = \alpha \cdot w_1 + \beta \cdot w_2 \\
v_2 = \gamma \cdot w_1 + \delta \cdot w_2
\]

and therefore

\[
\begin{bmatrix} a \\ b \end{bmatrix}_{B_1} = a \cdot v_1 + b \cdot v_2 = a \cdot (\alpha \cdot w_1 + \beta \cdot w_2) + b \cdot (\gamma \cdot w_1 + \delta \cdot w_2) = \left( \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right)_{B_2}
\]
Example 2.13. Let $B_1 = \{[1, 0], [0, 1]\}$ and $B_2 = \{[1, 1], [1, -1]\}$. To find $\Delta_{12}$ we form the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Row reduce until the left hand block is $I_2$, which is already the case. On the other hand, to find $\Delta_{21}$ we form the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

and row reduce, yielding the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{bmatrix}$$

A quick check verifies that indeed

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 2.14. Show that the change of basis algorithm allows us to find the inverse of an $n \times n$ matrix $A$ as follows: construct the $n \times 2n$ matrix $[A|I_n]$ and apply elementary row operations. If this results in a matrix $[I_n|B]$, then $B = A^{-1}$, if not, then $A$ is not invertible.

3. Diagonalization, Webpage Ranking, Data and Covariance

In this section, we develop the tools to analyze the smoking situation in Smallville. This will enable us to answer the questions posed earlier:

- What happens after $n$ years?
- Is there an equilibrium state?

The key idea is that a matrix represents a linear transformation with respect to a basis, so by choosing a different basis, we may get a “better” representation. Our goal is to take a square matrix $A$ and compute

$$\lim_{t \to \infty} A^t$$

So if “Tout est pour le mieux dans le meilleur des mondes possibles”, perhaps we can find a basis where $A$ is diagonal. Although Candide is doomed to disappointment, we are not! In many situations, we get lucky, and $A$ can be diagonalized. To tackle this, we switch from French to German.
3.1. Eigenvalues and Eigenvectors. Suppose 
\[ T : V \rightarrow V \]
is a linear transformation; for concreteness let \( V = \mathbb{R}^n \). If there exists a set of 
vectors \( B = \{v_1, \ldots, v_n\} \) with 
\[ T(v_i) = \lambda_i v_i, \text{ with } \lambda_i \in \mathbb{R}, \]
such that \( B \) is a basis for \( V \), then the matrix \( M_{BB} \) representing \( T \) with respect to 
\( B \) (which is a basis for both source and target of \( T \)) is of the form 
\[
M_{BB} = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & \lambda_n \\
\end{bmatrix}
\]
This is exactly what happened in Example 1.8, and our next task is to determine 
how to find such a lucky basis. Given a matrix \( A \) representing \( T \), we want to find 
vectors \( v \) and scalars \( \lambda \) satisfying 
\[ Av = \lambda v \] or equivalently \( (A - \lambda \cdot I_n) \cdot v = 0 \).
The kernel of a matrix \( M \) is the set of \( v \) such that \( M \cdot v = 0 \), so we need the kernel 
of \( (A - \lambda \cdot I_n) \). Since the determinant of a square matrix is zero exactly when 
the matrix has a nonzero kernel, this means we need to solve for \( \lambda \) in the equation 
\[ \det(A - \lambda \cdot I_n) = 0 \]. The corresponding solutions are the eigenvalues of the matrix \( A \).

**Example 3.1.** Let \( A \) be the matrix from Example 1.8:

\[
\begin{vmatrix}
7 - \lambda & 2 \\
3 & 8 - \lambda \\
\end{vmatrix} = (7 - \lambda)(8 - \lambda) - 6 = \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10).
\]

So the eigenvalues of \( A \) are 5 and 10. These are exactly the values that appear on 
the diagonal of the matrix \( D \); as we shall see, this is no accident.

For a given eigenvalue \( \lambda \), we must find the vector \( v \) which solves \((A - \lambda \cdot I)v = 0\); 
these vectors are the eigenvectors of \( A \). For this, we go back to solving systems of 
linear equations

**Example 3.2.** Staying in Smallville, we plug in our eigenvalues \( \lambda \in \{5, 10\} \). First 
we solve for \( \lambda = 5 \):

\[
\begin{bmatrix}
7 - 5 & 2 \\
3 & 8 - 5 \\
\end{bmatrix} \cdot v = 0
\]
which row reduces to the system

\[
\begin{bmatrix}
1 & 1 \\
0 & 0 \\
\end{bmatrix} \cdot \mathbf{v} = 0,
\]

which has as solution any nonzero multiple of \([1, -1]^T\). A similar computation for \(\lambda = 10\) yields the eigenvector \([2, 3]^T\).

We’ve mentioned that the eigenvalues appear on the diagonal of the matrix \(D\). Go back and take a look at Example 1.8 and see if you can spot how the eigenvectors come into play. If you get stuck, no worries: we tackle this next.

### 3.2. Diagonalization.

We now revisit the change of basis construction. Let \(T : V \rightarrow V\) be a linear transformation, and suppose we have two bases \(B_1\) and \(B_2\) for \(V\). What is the relation between 

\[ M_{B_1B_1} \text{ and } M_{B_2B_2}. \]

The matrix \(M_{B_1B_1}\) takes as input a vector \(\mathbf{v}_{B_1}\) written in terms of the \(B_1\) basis, applies the operation \(T\), and outputs the result in terms of the \(B_1\) basis. We described change of basis as analogous to translation. To continue with this analogy, suppose \(T\) represents a recipe, \(B_1\) is French and \(B_2\) is German. Chef Pierre is French, and diner Hans is German. So Hans places his order \(v_{B_2}\) in German. Chef Pierre is temperamental—the waiter dare not pass on an order in German—so the order must be translated to French:

\[ \mathbf{v}_{B_1} = \Delta_{12} \mathbf{v}_{B_2}. \]

This is relayed to Pierre, who produces

\[ (M_{B_1B_1}) \cdot \Delta_{12} \mathbf{v}_{B_2}. \]

Alas, Hans also has a short fuse, so the beleaguered waiter needs to present the dish to Hans with a description in German, resulting in

\[ (\Delta_{21}) \cdot (M_{B_1B_1}) \cdot (\Delta_{12}) \mathbf{v}_{B_2}. \]

Et voila! Comity in the restaurant. The reader unhappy with culinary analogies (or levity) should ignore the verbiage above, but keep the formulas.

**Example 3.3.** Let \(B_1 = \{[1, 0], [0, 1]\}\) be the standard basis for \(\mathbb{R}^2\), and \(B_2 = \{[1, -1], [2, 3]\}\) be the basis of eigenvectors we computed in Example 3.2. Using the algorithm of Exercise 2.14, we have that

\[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
-1 & 3 & 0 & 1 \\
\end{bmatrix}
\]

row reduces to

\[
\begin{bmatrix}
1 & 0 & 3/5 & -2/5 \\
0 & 1 & 1/5 & 1/5 \\
\end{bmatrix}
\]

and a check shows that

\[
\begin{bmatrix}
3/5 & -2/5 \\
1/5 & 1/5 \\
\end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = I_2.
\]
These are exactly the matrices $B$ and $B^{-1}$ which appear in Example 1.8. Let’s check our computation:

\[
\begin{bmatrix}
\frac{3}{5} & \frac{-2}{5} \\
\frac{1}{5} & \frac{1}{5}
\end{bmatrix} \cdot \begin{bmatrix}
7 & 2 \\
3 & 8
\end{bmatrix} \cdot \begin{bmatrix}
1 & 2 \\
-1 & 3
\end{bmatrix} = \begin{bmatrix}
5 & 0 \\
0 & 10
\end{bmatrix}
\]

So we find

\[
\lim_{t \to \infty} \begin{bmatrix}
\frac{7}{3} & \frac{2}{3} \\
\frac{3}{8} & \frac{1}{8}
\end{bmatrix}^t = \begin{bmatrix}
1 & 2 \\
-1 & 3
\end{bmatrix} \cdot \left( \lim_{t \to \infty} \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & 1
\end{bmatrix}^t \right) \cdot \begin{bmatrix}
\frac{3}{5} & \frac{-2}{5} \\
\frac{1}{5} & \frac{1}{5}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 2 \\
-1 & 3
\end{bmatrix} \cdot \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\frac{3}{5} & \frac{-2}{5} \\
\frac{1}{5} & \frac{1}{5}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
.4 & .4 \\
.6 & .6
\end{bmatrix}
\]

Multiplying the last matrix by our start state vector $[n(0), s(0)]^T = [2000, 8000]^T$, we find the equilibrium state is $[4000, 6000]^T$. It is interesting that although we began with far more smokers than nonsmokers, and even though every year the percentage of nonsmokers who began smoking was larger than the percentage of smokers who quit, nevertheless in the equilibrium state we have more nonsmokers than in the initial state.

**Exercise 3.4.** Show that the matrix

\[
\begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

which rotates a vector in $\mathbb{R}^2$ counterclockwise by $\theta$ degrees has real eigenvalues only when $\theta = 0$ or $\theta = \pi$. □

### 3.3. Ranking using diagonalization.

Diagonalization is the key tool in many web search engines. The first task is to determine the right structure to represent the web; we will use a weighted, directed graph. Vertices of the graph correspond to websites, and edges correspond to links. If website A has a link to website B, this is represented by a directed edge from vertex A to vertex B; if website A has $k$ links to other pages, each directed edge is assigned weight $\frac{1}{k}$. The idea is that a browser viewing website A has (in the absence of other information) an equal chance of choosing to click on any of the $k$ links.
From the data of a weighted, directed graph on vertices \( \{v_1, \ldots, v_n\} \) we construct an \( n \times n \) matrix \( T \). Let \( l_j \) be the number of links at vertex \( v_j \). Then

\[
T_{ij} = \begin{cases} 
\frac{1}{l_j} & \text{if vertex } j \text{ has a link to vertex } i \\
0 & \text{if vertex } j \text{ has no link to vertex } i.
\end{cases}
\]

**Example 3.5.** Consider the graph \( \Gamma \):

![Graph \( \Gamma \)](image)

Using \( \Gamma \), we construct the matrix

\[
T = \begin{bmatrix}
0 & 1/2 & 1/3 & 0 \\
0 & 0 & 1/3 & 0 \\
1/2 & 0 & 0 & 1 \\
1/2 & 1/2 & 1/3 & 0
\end{bmatrix}.
\]

The matrix \( T \) is **column stochastic**: all column sums are one, which reflects the fact that the total probability at each vertex is one. There is a subtle point which makes this different than the Smallville smoking situation: a user may not choose to click on a link on the current page, but type in a URL. The solution is to add a second matrix \( R \) that represents the possibility of a **random jump**. Assuming that the user is equally likely to choose any of the \( n \) websites possible, this means that \( R \) is an \( n \times n \) matrix with all entries \( 1/n \). Putting everything together, we have

\[
G = (1 - p) \cdot T + p \cdot R,
\]
where \( p \) is the probability of a random jump.

The matrix \( G \) is called the **Google matrix**; experimental evidence indicates that \( p \) is close to .15. If a user is equally likely to start at any of the \( n \) vertices representing websites, the initial input vector to \( G \) is \( \mathbf{v}(0) = [1/n, \ldots, 1/n]^T \). As in our previous vignette, we want to find

\[
\mathbf{v}_\infty = \left( \lim_{t \to \infty} G^t \right) \cdot \mathbf{v}(0)
\]

**Theorem 3.6.** [Perron-Frobenius] If \( M \) is a column stochastic matrix with all entries positive, then 1 is an eigenvalue for \( M \), and the corresponding eigenvector \( \mathbf{v}_\infty \) has all positive entries, which sum to one. The limit above converges to \( \mathbf{v}_\infty \).

Therefore, one way of ranking webpages reduces to solving

\[
G \cdot \mathbf{v} = I_n \cdot \mathbf{v}
\]

The problem is that \( n \) is in the billions. In practice, rather than solving for \( \mathbf{v} \) it is easier to get a rough approximation to \( \mathbf{v}_\infty \) from \( G^m \cdot \mathbf{v}(0) \) for \( m \) not too big.
3.4. Data Application: Diagonalization of the Covariance Matrix. In this section, we discuss an application of diagonalization in statistics. Suppose we have a data sample \( X = \{p_1, \ldots, p_k\} \), with the points \( p_i = (p_{i1}, \ldots, p_{im}) \in \mathbb{R}^m \). How do we visualize the spread of the data? Is it concentrated in certain directions? Do large subsets cluster? Linear algebra provides one way to attack the problem.

First, we need to define the covariance matrix: let \( \mu_j(X) \) denote the mean of the \( j^{th} \) coordinate of the points of \( X \), and form the matrix

\[
N_{ij} = p_{ij} - \mu_j(X)
\]

The matrix \( N \) represents the original data, but with the points translated with respect to the mean in each coordinate.

**Definition 3.7.** The covariance matrix of \( X \) is \( N^T \cdot N \).

**Example 3.8.** Consider the dataset

\[
X = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4)\}
\]

Since \( \sum p_{i1} = 15 = \sum p_{i2} \), we have \( \mu_1(X) = 2.5 = \mu_2(X) \), so

\[
N = \frac{1}{2} \begin{bmatrix}
-3 & -3 \\
-1 & -1 \\
-1 & 1 \\
1 & -1 \\
1 & 1 \\
3 & 3
\end{bmatrix}
\]

The covariance matrix is therefore

\[
N^T \cdot N = \frac{1}{4} \begin{bmatrix}
-3 & -1 & -1 & 1 & 1 & 3 \\
-3 & -1 & 1 & -1 & 1 & 3
\end{bmatrix} \cdot \begin{bmatrix}
-3 & -3 \\
-1 & -1 \\
1 & 1 \\
1 & 1 \\
3 & 3
\end{bmatrix} = \begin{bmatrix}
11/2 & 9/2 \\
9/2 & 11/2
\end{bmatrix}
\]

To find the eigenvalues and eigenvectors for the covariance matrix, we compute:

\[
\det \begin{bmatrix}
11/2 - \lambda & 9/2 \\
9/2 & 11/2 - \lambda
\end{bmatrix} = \lambda^2 - 11\lambda + 10 = (\lambda - 1)(\lambda - 10).
\]
Exercise 3.9. Show when $\lambda = 1$, $v = [1, -1]^T$ and when $\lambda = 10$, $v = [1, 1]^T$. 

Theorem 3.10. Order the eigenvectors $\{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m\}$. The data varies in proportion to the eigenvalues, in the direction of the associated eigenvector.

The punchline is that the biggest eigenvalue corresponds to the biggest variance of the data, which “spreads out” in the direction of the corresponding eigenvector. Example 3.8 illustrates this nicely: the eigenvalue 10 corresponds to the direction $[1, 1]$, where the data spreads out the most, and the eigenvalue 1 corresponds to the direction $[1, -1]$, which is the second largest direction of spread. So for two dimensional data, the ellipse which best approximates the data is determined by the eigenvectors and eigenvalues of the covariance matrix. Theorem 3.10 is the starting point of principal component analysis, which is a staple of applied mathematics.

4. Orthogonality, Least Squares Fitting, Singular Value Decomposition

The concept of orthogonality plays a key role in data science—generally it is not possible to perfectly fit data to reality, so the focus is on approximations. We want a good approximation, which will entail minimizing the distance between our approximation and the exact answer. The squared distance between two points is a quadratic function, so taking derivatives to minimize distance results in a system of linear equations. Vector calculus teaches us that minimization problems often involve projection onto a subspace. We now examine two fundamental tools in data analysis: least squares data fitting, and singular value decomposition. We start with a warm-up exercise on the law of cosines: for a triangle with side lengths $A, B, C$ and opposite angles $a, b, c$, the law of cosines is

$$A^2 = B^2 + C^2 - 2AB \cos(c)$$

Exercise 4.1. Justify the assertion made in Definition 1.4 that the dot product is zero when vectors are orthogonal, as follows. Let the roles of $A, B, C$ in the law of cosines be played by the lengths of vectors $v, w, w - v$, with $v$ and $w$ both emanating from the origin and $c$ the angle between them. Apply the law of cosines to show that

$$v \cdot w = |v| \cdot |w| \cdot \cos(c).$$

Since $\cos(c) = 0$ only when $c \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}$, the result follows.

4.1. Least Squares. Let $X = \{p_1, \ldots, p_n\}$ be a set of data points in $\mathbb{R}^m$, and suppose we want to fit a curve (if $m = 2$) or a surface (if $m = 3$) or some other geometric structure to the data. If we allow too much freedom for our geometric object, the result is usually not a good approximation: for example, if $m = 2$ then since the space of polynomials of degree at most $k$ has dimension $\binom{k+2}{2}$, there is a polynomial $f(x, y)$ such that $f(p_i) = 0$ for all $p_i \in X$ as soon as $\binom{k+2}{2} > n$. However, this polynomial often has lots of oscillation away from the points of $X$. 
So we need to make the question more precise, and develop criteria to evaluate what makes a fit “good”.

**Example 4.2.** Consider a data set consisting of points in $\mathbb{R}^2$

$$X = \{(1, 6), (2, 5), (3, 7), (4, 10)\}$$

What line best approximates the data? Since a line will be given by the equation $y = ax + b$, the total error in the approximation will be

$$\text{error} = \sqrt{\sum_{(x_i, y_i) \in X} (y_i - (ax_i + b))^2}$$

For this example, the numbers are

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$ax + b$</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>$a + b$</td>
<td>$(6 - (a + b))^2$</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>$2a + b$</td>
<td>$(5 - (2a + b))^2$</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>$3a + b$</td>
<td>$(7 - (3a + b))^2$</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>$4a + b$</td>
<td>$(10 - (4a + b))^2$</td>
</tr>
</tbody>
</table>

Minimizing $\sqrt{f}$ is equivalent to minimizing $f$, so we need to minimize

$$f(a, b) = (6 - (a + b))^2 + (5 - (2a + b))^2 + (7 - (3a + b))^2 + (10 - (4a + b))^2 = 30a^2 + 20ab + 4b^2 - 154a - 56b + 210$$

which has partials

$$\frac{\partial f}{\partial a} = 60a + 20b - 154$$

$$\frac{\partial f}{\partial b} = 8b + 20a - 56$$

yielding $b = 3.5, a = 1.4$. However, there is another way to look at this problem: we want to find the best solution to the system

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \\ 10 \end{bmatrix}$$

This is equivalent to asking for the vector in the span of the columns of the left hand matrix which is closest to $[6, 5, 7, 10]^T$.

To understand how the approach to minimizing using partials relates to finding the closest vector in a subspace, we need to look more deeply at orthogonality.

### 4.2. Subspaces and orthogonality.

**Definition 4.3.** A subspace $V'$ of a vector space $V$ is a subset of $V$, which is itself a vector space. Subspaces $W$ and $W'$ of $V$ are orthogonal if

$$w \cdot w' = 0 \text{ for all } w \in W, w' \in W'$$
A set of vectors \( \{v_1, \ldots, v_n\} \) is orthonormal if \( v_i \cdot v_j = \delta_{ij} \) (1 if \( i = j \), 0 otherwise). A matrix \( A \) is orthonormal when the column vectors of \( A \) are orthonormal.

**Example 4.4.** Examples of subspaces. Let \( A \) be an \( m \times n \) matrix.

(a) The row space \( R(A) \) of \( A \) is the span of the row vectors of \( A \).

(b) The column space \( C(A) \) of \( A \) is the span of the column vectors of \( A \).

(c) The null space \( N(A) \) of \( A \) is the set of solutions to the system \( A \cdot x = 0 \).

(d) If \( W \) is a subspace of \( V \), then \( W^\perp = \{v | v \cdot w = 0 \text{ for all } w \in W \} \).

Notice that \( v \in N(A) \text{ iff } v \cdot w = 0 \) for every row vector of \( A \).

**Exercise 4.5.** Properties of important subspaces for \( A \) as above.

(a) Prove \( N(A) = R(A)^\perp \) and \( N(A^T) = C(A)^\perp \).

(b) Prove any vector \( v \in V \) has a unique decomposition as \( v = w' + w'' \) with \( w' \in W \) and \( w'' \in W^\perp \).

(c) Prove that for a vector \( v \in V \), the vector \( w \in W \) which minimizes \( |v - w| \) is the vector \( w' \) above.

The rank of \( A \) is \( \dim C(A) = \dim R(A) \). Show \( n = \dim N(A) + \dim R(A) \). ♦

Returning to the task at hand, the goal is to find the \( x \) that minimizes \( |Ax - b| \), which is equivalent to finding the vector in \( C(A) \) closest to \( b \). By Exercise 4.5, we can write \( b \) uniquely as \( b' + b'' \) with \( b' \in C(A) \) and \( b'' \in C(A)^\perp \). Continuing to reap the benefits of Exercise 4.5, we have

\[
b = b' + b'' \Rightarrow A^T b = A^T b' + A^T b'' = A^T A \cdot y + 0,
\]

where the last equality follows because \( b' \in C(A) \Rightarrow b' = A \cdot y \), and since \( b'' \in C(A)^\perp \Rightarrow A^T b'' = 0 \). Thus

\[
A^T b = A^T A \cdot y \text{ and therefore } y = (A^T A)^{-1} A^T b
\]

solves the problem, as long as \( A^T A \) is invertible. In Exercise 1.7 we showed that \( A^T A \) is symmetric; it turns out that symmetric matrices are exactly those which have the property that they can be diagonalized by an orthonormal change of basis matrix \( B \) (notice that this means \( B^T = B^{-1} \)). This fact is called the spectral theorem. One direction is easy: suppose \( A \) is a matrix having such a change of basis. Then

\[
BAB^{-1} = D = D^T = (BAB^{-1})^T = BA^T B^{-1} \Rightarrow A = A^T
\]

**Exercise 4.6.** Show that a real symmetric matrix has only real eigenvalues, and this need not hold for an arbitrary real matrix. Now use induction to prove that if \( A \) is symmetric, it admits an orthonormal change of basis matrix as above. ♦
Example 4.7. Let $A$ denote the matrix on the left in the last displayed equation in Example 4.2, and let $b = [6, 5, 7, 10]^T$. Then

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

so

$$(A^T A)^{-1} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{bmatrix}$$

Continuing with the computation, we have

$$A \cdot (A^T A)^{-1} \cdot A^T = \frac{1}{10} \begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix}$$

Putting everything together, we see that indeed

$$A \cdot (A^T A)^{-1} \cdot A^T \cdot b = \begin{bmatrix} 4.9 \\ 6.3 \\ 7.7 \\ 9.1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3.5 \\ 1.4 \end{bmatrix}$$

where $[3.5, 1.4]$ is the solution we obtained using partials.

Exercise 4.8. What happens when we fit a degree two equation $y = ax^2 + bx + c$ to the data of Example 4.2? The corresponding matrix equation is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \cdot \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \\ 10 \end{bmatrix}$$

Carry out the analysis conducted in Example 4.7 for this case, and show that the solution obtained by solving for $y$ and computing $b^T = Ay$ agrees with the solution obtained by using partials. ◊

4.3. Singular Value Decomposition. Diagonalization can provide elegant and computationally efficient solutions to questions like that posed in Example 1.1. One drawback is that we are constrained to square matrices, whereas the problems encountered in the real world often have the target different from the source, so that the matrix or linear transformation in question is not square. Singular Value Decomposition is “diagonalization for non-square matrices”.
Theorem 4.9. Let $M$ be an $m \times n$ matrix of rank $r$. Then there exist matrices $U$ of size $m \times m$ and $V$ of size $n \times n$ with orthonormal columns, and $\Sigma$ an $m \times n$ diagonal matrix with nonzero entries $\Sigma_{ii} = \{\sigma_1, \ldots, \sigma_r\}$, such that

$$M = U\Sigma V^T.$$ 

Proof. The matrix $M^T M$ is symmetric, so by Exercise 4.6 there is an orthonormal change of basis that diagonalizes $M^T M$. Let $M^T M \cdot v_j = \lambda_j v_j$, and note that

$$v_i^T M^T M v_j = \lambda_j v_i^T v_j = \lambda_j \delta_{ij}.$$ 

Define $\sigma_i = \sqrt{\lambda_i}$, and $q_i = \frac{1}{\sigma_i} M v_i$. Then

$$q_i^T q_j = \delta_{ij} \text{ for } j \in \{1, \ldots, r\}.$$ 

Extend the $q_i$ to a basis for $\mathbb{R}^m$, and let $U$ be the matrix whose columns are the $q_i$, and $V$ the matrix whose columns are the $v_i$. Then

$$(U^T M V)_{ij} = q_i^T (M V)_{col} = q_i^T M v_j = \sigma_j q_i^T q_j = \sigma_j \delta_{ij} \text{ if } j \leq r, \text{ or } 0 \text{ if } j > r.$$ 

Hence $U^T M V = \Sigma$; the result follows using that $U^T = U^{-1}$ and $V^T = V^{-1}$. □

Example 4.10. We compute the Singular Value Decomposition for

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

First, we compute

$$M^T M = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

We find (do it!) that the eigenvalues are $\{7, 3\}$ with corresponding eigenvectors $\{(1, 1)^T, (1, -1)^T\}$. The eigenvectors are the columns of $V^T$; they both have length $\sqrt{2}$ so to make them orthonormal we scale by $\frac{1}{\sqrt{2}}$, and we have

$$\Sigma = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \text{ and } V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

It remains to calculate $U$, which we do using the recipe $u_i = \frac{1}{\sigma_i} \cdot M v_i$. This gives us

$$u_1 = \frac{1}{\sqrt{14}} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } u_2 = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$
4. Orthogonality, Least Squares Fitting, Singular Value Decomposition

We need to extend this to an orthonormal basis for \( \mathbb{R}^3 \), so we need to calculate a basis for \( N(M^T) = C(M)^\perp \), which we find consists of \( u_3 = \frac{1}{\sqrt{21}}[-2, 4, 1]^T \), hence

\[
U = \begin{bmatrix}
\frac{3}{\sqrt{14}} & -1 & \frac{-2}{\sqrt{21}} \\
\frac{1}{\sqrt{14}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{21}} \\
\frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{21}}
\end{bmatrix}
\]

Exercise 4.11. Check that indeed \( M = U\Sigma V^T \). What is the rank one matrix which best approximates \( M \)?

The utility of SVD stems from the fact that it allows us to represent the matrix \( M \) as a sum of simpler matrices, in particular, matrices of rank one.

Example 4.12.

\[
M = u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \cdots + u_r\sigma_r v_r^T
\]

which means we can decompose \( M \) as a sum of rank one matrices; the bigger the value of \( \sigma_i \), the larger the contribution to \( M \). For example, a greyscale image is comprised of \( m \times n \) pixels, and by doing an SVD decomposition, we can get a good approximation to the image which takes up very little storage by keeping only the highest weight terms.

Exercise 4.13. Least squares approximation is also an instance of SVD. Recall that in least squares, we are trying to find the best approximate solution to a system \( Mx = b \). So our goal is to minimize \( |Mx - b| \). We have

\[
|Mx - b| = |U\Sigma V^T x - b| = |\Sigma V^T x - U^T b| = |\Sigma y - U^T b|
\]

Show this is minimized by

\[
y = V^T x = \frac{1}{\Sigma} U^T b,
\]

where \( \frac{1}{\Sigma} \) is a matrix with \( \frac{1}{\sigma_i} \) on the diagonal.
Groups, Rings, Modules

The previous chapter covered linear algebra, and in this chapter we move on to more advanced topics in abstract algebra, starting with the concepts of group and ring. In Chapter 1, we defined a vector space over a field \( \mathbb{K} \) without giving a formal definition for a field; this is rectified in §1. A field is a specific type of ring, so we will define a field in the context of a more general object. The reader has already encountered many rings besides fields: the integers are a ring, as are polynomials in one or more variables, and square matrices. When we move into the realm of topology, we’ll encounter more exotic rings, such as differential forms. Rings bring a greater level of complexity into the picture, and with that, the ability to build structures and analyze objects in finer detail. In this chapter, we’ll cover

- Groups, Rings and Homomorphisms.
- Modules and Operations on Modules.
- Localization of Rings and Modules.
- Noetherian Rings, Hilbert Basis Theorem, Variety of an Ideal.

1. Groups, Rings and Homomorphisms

1.1. Groups. Let \( G \) be a set of elements endowed with a binary operation sending \( G \times G \to G \) via \( (a, b) \mapsto a \cdot b \); in particular \( G \) is closed under the operation. The set \( G \) is a group if the following three properties hold:

- \( \cdot \) is associative: \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \).
- \( G \) possesses an identity element \( e \) such that \( \forall g \in G, \ g \cdot e = e \cdot g = g \).
- Every \( g \in G \) has an inverse \( g^{-1} \) such that \( g \cdot g^{-1} = g^{-1} \cdot g = e \).

The group operation is commutative if \( a \cdot b = b \cdot a \); in this case the group is abelian. The prototypical example of an abelian group are the set of all integers,
with addition serving as the group operation. For abelian groups it is common to write the group operation as $+$. In a similar spirit,

$$\mathbb{Z}/n\mathbb{Z}$$

(often written $\mathbb{Z}/n$ for brevity)

is an abelian group with group operation addition modulo $n$. If instead of addition in $\mathbb{Z}$ we attempt to use multiplication as the group operation, we run into a roadblock: zero clearly has no inverse.

For an example of a group which is not abelian, recall that matrix multiplication is not commutative. The set $G$ of $n \times n$ invertible matrices with entries in $\mathbb{R}$ is a non-abelian group if $n \geq 2$, with group operation matrix multiplication.

**Exercise 1.1.** Determine the multiplication table for the group of $2 \times 2$ invertible matrices with entries in $\mathbb{Z}/2$. There are $16$ $2 \times 2$ matrices with $\mathbb{Z}/2$ entries, but any matrix of rank zero or rank one is not a member. You should find $6$ elements. $
$

**Definition 1.2.** A **subgroup** of a group $G$ is a subset of $G$ which is itself a group. A subgroup $H$ of $G$ is **normal** if $gHg^{-1} \subseteq H$ for all $g \in G$. A **homomorphism** of groups is a map which preserves the group structure, so a map $G_1 \xrightarrow{f} G_2$ is a homomorphism if

$$f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$$

for all $g_i \in G_i$.

The kernel of $f$ is the set of $g \in G_1$ such that $f(g) = e$. If the kernel of $f$ is $\{e\}$ then $f$ is injective or one-to-one; if every $g \in G_2$ has $g = f(g')$ for some $g' \in G_1$ then $f$ is surjective or onto, and if $f$ is both injective and surjective, then $f$ is an isomorphism.

**Exercise 1.3.** Prove that the kernel of a homomorphism is a normal subgroup. Prove that the condition $gHg^{-1} \subseteq H$ that defines a normal subgroup insures that the quotient

$$G/H = \text{equivalence classes } a \sim b \text{ iff } aH = bH \text{ iff } ab^{-1} \in H.$$  

Then $G/H$ is itself a group. $
$

**Exercise 1.4.** Label the vertices of an equilateral triangle as $\{1, 2, 3\}$, and consider the rigid motions, which are rotations by integer multiples of $\frac{2\pi}{3}$ and reflection about any line connecting a vertex to the midpoint of the opposite edge. Prove that this group has $6$ elements, and is isomorphic to the group in Exercise 1.1, as well as to the group $S_3$ of permutations of three letters, with the group operation in $S_3$ given by composition: if $\sigma(1) = 2$ and $\tau(2) = 3$, then $\tau \cdot \sigma(1) = 3$.

**Definition 1.5.** There are a number of standard operations on groups:

- Direct Sum.
- Intersection.
- Sum (when Abelian).
The direct sum $G \oplus G'$ of groups $G$ and $G'$ consists of pairs $(g, g')$ with $g \in G$, $g' \in G'$, with group operation defined pointwise
\[(g, g') \cdot (h, h') = (gh, g'h').\]
To form intersection and sum, we need to have groups $G$ and $G'$ existing as subgroups of a larger ambient group $G''$; this condition allows us to form the sets
\[G \cap G' = \{g \mid g \in G \text{ and } g \in G'\}\]
\[G + G' = \{g + g' \text{ for some } g \in G, g' \in G'\}\]
For the sum $G + G'$ we write the group operation additively; there is a version of sum when $G$ is not Abelian, but we shall not need it.

Exercise 1.6. Show the constructions in the previous definition yield groups.

1.2. Rings. A ring $R$ is an abelian group under addition (which henceforth will always be written as $+$, with additive identity written as $0$), with an additional associative operation multiplication ($\cdot$) which is distributive with respect to addition. An additive subgroup $I \subseteq R$ such that $r \cdot i \in I$ for all $r \in R, i \in I$ is an ideal. In these notes, unless otherwise noted, rings will have
- a multiplicative identity, written as $1$.
- commutative multiplication.

Example 1.7. Examples of rings not satisfying the above properties:
- As a subgroup of $\mathbb{Z}$, the even integers $2\mathbb{Z}$ satisfy the conditions above for the usual multiplication and addition, but have no multiplicative identity.
- The set of all $2 \times 2$ matrices over $\mathbb{R}$ is an abelian group under $+$, has a multiplicative identity, and satisfies associativity and distributivity. But multiplication is not commutative.

Definition 1.8. A field is a commutative ring with unit, such that every nonzero element has a multiplicative inverse. A nonzero element $a$ of a ring is a zero divisor if there is a nonzero element $b$ with $a \cdot b = 0$. An integral domain (for brevity, domain) is a ring with no zero divisors.

Remark 1.9. General mathematical culture: a noncommutative ring such that every nonzero element has a left and right inverse is called a division ring. The most famous example is the ring of quaternions, discovered by Hamilton in 1843 and etched into the Brougham Bridge.

Example 1.10. Examples of rings.
- $\mathbb{Z}$, the integers, and $\mathbb{Z}/n\mathbb{Z}$, the integers mod $n$.
- $A[x_1, \ldots, x_n]$, the polynomials with coefficients in a ring $A$.
- $C^0(\mathbb{R})$, the continuous functions on $\mathbb{R}$.
- $\mathbb{K}$ a field.
**Definition 1.11.** If $R$ and $S$ are rings, a map $\phi : R \to S$ is a ring homomorphism if it respects the ring operations: for all $r, r' \in R$,

(a) $\phi(r \cdot r') = \phi(r) \cdot \phi(r')$.
(b) $\phi(r + r') = \phi(r) + \phi(r')$.
(c) $\phi(1) = 1$.

**Example 1.12.** There is no ring homomorphism (other than the zero map) from $\mathbb{Z}/2 \xrightarrow{\phi} \mathbb{Z}$.

To see this, note that in $\mathbb{Z}/2$ the zero element is 2. Hence in $\mathbb{Z}$ we would have $0 = \phi(2) = \phi(1) + \phi(1) = 1 + 1 = 2$

An important construction is that of the **quotient ring**:

**Definition 1.13.** Let $I \subseteq R$ be an ideal. Elements of the quotient ring $R/I$ are cosets of the form $r + [I]$, with operations computed modulo $I$.

$$(r + [I]) \cdot (r' + [I]) = r \cdot r' + [I]$$

$$(r + [I]) + (r' + [I]) = r + r' + [I]$$

Note a contrast with the construction of a quotient group: for $H$ a subgroup of $G$, $G/H$ is itself a group only when $H$ is a normal subgroup. There is not a similar constraint on a ring quotient, because the additive operation in a ring is commutative, so all additive subgroups of $R$ (in particular, ideals) are abelian, hence are normal subgroups with respect to the additive structure.

**Exercise 1.14.** Prove the kernel of a ring map is an ideal. $\diamond$

2. **Modules and Operations on Modules**

In linear algebra, we can add two vectors together, or multiply a vector by an element of the field over which the vector space is defined. Module is to ring what vector space is to field. We saw above that a field is a special type of ring; a module over a field is a vector space.

**Definition 2.1.** A module $M$ over a ring $R$ is an abelian group, together with an action of $R$ on $M$ which is $R$-linear: for $r_i \in R$, $m_i \in M$,

- $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$,
- $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$,
- $r_1(r_2m_1) = (r_1r_2)m_1$,
- $1 \cdot (m) = m$.
- $0 \cdot (m) = 0$.
- $0 + m = m$. 
An $R$-module $M$ is finitely-generated if there exist $\{m_1, \ldots, m_n\} \subseteq M$ such that any $m \in M$ can be written

$$m = \sum_{i=1}^{n} r_i m_i$$

for some $r_i \in R$. A submodule $I \subseteq R$ is an ideal.

**Exercise 2.2.** Which are ideals?

(a) $\{ f \in C^0(\mathbb{R}) \mid f(1) = 0 \}$.

(b) $\{ f \in C^0(\mathbb{R}) \mid f(1) \neq 0 \}$.

(c) $\{ n \in \mathbb{Z} \mid n = 0 \mod 3 \}$.

(d) $\{ n \in \mathbb{Z} \mid n \neq 0 \mod 3 \}$.

**Example 2.3.** Examples of modules over a ring $R$.

(a) Any ring is a module over itself.

(b) A quotient ring $R/I$ is both an $R$-module and an $R/I$-module.

(c) An $R$-module $M$ is free if it is free of relations among a minimal set of generators, so a finitely generated free $R$-module is isomorphic to $R^n$. An ideal with more than two generators is not free: if $I = \langle f, g \rangle$ then we have the trivial (or Koszul) relation $f \cdot g - g \cdot f = 0$ on $I$. For modules $M$ and $N$, Definition 1.5 shows we can construct a new module via the direct sum $M \oplus N$. If $M, N \subseteq P$ we also have modules $M \cap N$ and $M + N$.

**Definition 2.4.** Let $M_1$ and $M_2$ be modules over $R$, $m_i \in M_i$, $r \in R$. A homomorphism of $R$-modules $\psi : M_1 \to M_2$ is a function $\psi$ such that

(a) $\psi(m_1 + m_2) = \psi(m_1) + \psi(m_2)$.

(b) $\psi(r \cdot m_1) = r \cdot \psi(m_1)$.

Notice that when $R = \mathbb{K}$, these two conditions are exactly those for a linear transformation, again highlighting that module is to ring as vector space is to field.

**Definition 2.5.** Let $M_1 \xrightarrow{\phi} M_2$ be a homomorphism of $R$-modules.

- The kernel of $\phi$ consists of those $m_1 \in M_1$ such that $\phi(m_1) = 0$.

- The image of $\phi$ consists of those $m_2 \in M_2$ such that $m_2 = \phi(m_1)$ for some $m_1 \in M_1$.

- The cokernel of $\phi$ consists of $M_2/\text{im}(M_1)$.

**Exercise 2.6.** Prove that the kernel, image, and cokernel of a homomorphism of $R$-modules are all $R$-modules.
A sequence of $R$–modules and homomorphisms

$$
\mathcal{C}: \cdots \xrightarrow{\phi_{j+2}} M_{j+1} \xrightarrow{\phi_{j+1}} M_j \xrightarrow{\phi_j} M_{j-1} \xrightarrow{\phi_{j-1}} \cdots
$$

is a complex (or chain complex) if

$$\text{im}(\phi_{j+1}) \subseteq \ker(\phi_j).$$

The sequence is exact at $M_j$ if $\text{im}(\phi_{j+1}) = \ker(\phi_j)$; a complex which is exact everywhere is called an exact sequence. The $j^{th}$ homology module of $\mathcal{C}$ is:

$$H_j(\mathcal{C}) = \ker(\phi_j)/\text{im}(\phi_{j+1}).$$

An exact sequence of the form

$$\mathcal{C}: 0 \rightarrow A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \rightarrow 0$$

is called a short exact sequence.

**Exercise 2.7.** This exercise foreshadows the key concept of algebraic topology, which is to use algebra to encode topological features of a space. Consider a trio of vector spaces over $\mathbb{K} = \mathbb{Z}/2$, of dimensions $\{1, 3, 3\}$. Let $\{012\}$ be a basis for $A_2$, $\{[01], [02], [12]\}$ a basis for $A_1$, and $\{[0], [1], [2]\}$ a basis for $A_0$. The labeling of basis elements is prompted by the pictures below; the map $d_i$ represents the boundaries of elements of $A_i$.

Show that if $d_2 = [1, 1, 1]^T$ and

$$d_1 = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}$$

then the sequence 2.1 is exact except at $A_0$. Use this to show the complex below

$$0 \rightarrow A_1 \xrightarrow{d_1} A_0 \rightarrow 0$$

has $H_0 \simeq \mathbb{K} \simeq H_1$. We will return to this example in Chapter 5; roughly speaking the nonzero $H_1$ reflects the fact that a (hollow) triangle is topologically the same as $S^1$, which has a one dimensional “hole”. In Exercise 2.7 of Chapter 5, you’ll prove that $H_0 \simeq \mathbb{K}$ reflects the fact that $S^1$ is connected. $\diamond$
2. Modules and Operations on Modules

2.1. Ideals. The most commonly encountered modules are ideals, which are submodules of the ambient ring $R$. An ideal $I \subseteq R$ is proper if $I \neq R$.

**Definition 2.8.** Types of proper ideals $I$

(a) $I$ is principal if $I$ can be generated by a single element.
(b) $I$ is prime if $f \cdot g \in I$ implies either $f$ or $g$ is in $I$.
(c) $I$ is maximal if there is no proper ideal $J$ with $I \subset J$.
(d) $I$ is primary if $f \cdot g \in I \Rightarrow f$ or $g^m$ is in $I$, for some $m \in \mathbb{N}$.
(e) $I$ is reducible if there exist ideals $J_1, J_2$ such that $I = J_1 \cap J_2$, $I \subset J_i$.
(f) $I$ is radical if $f^m \in I$ ($m \in \mathbb{N} = \mathbb{Z}_{>0}$) implies $f \in I$.

**Exercise 2.9.** Which classes above do the ideals $I \subseteq R[x, y]$ below belong to?

(a) $\langle xy \rangle$
(b) $\langle y - x^2, y - 1 \rangle$
(c) $\langle y, x^2 - 1, x^5 - 1 \rangle$
(d) $\langle y - x^2, y^2 - yx^2 + xy - x^3 \rangle$
(e) $\langle xy, x^2 \rangle$

Hint: draw a picture of corresponding solution set in $\mathbb{R}^2$.

If $I \subseteq R$ is an ideal, then properties of $I$ are often reflected in the structure of the quotient ring $R/I$.

**Theorem 2.10.**

$R/I$ is a domain $\iff$ $I$ is a prime ideal.

$R/I$ is a field $\iff$ $I$ is a maximal ideal.

**Proof.** For the first part, $R/I$ is a domain iff there are no zero divisors, hence $a \cdot b = 0$ implies $a = 0$ or $b = 0$. If $\bar{a}$ and $\bar{b}$ are representatives in $R$ of $a$ and $b$, then $a \cdot b = 0$ in $R/I$ is equivalent to

$$\bar{a} \bar{b} \in I \iff \bar{a} \in I \text{ or } \bar{b} \in I,$$

which holds iff $I$ is prime. For the second part, suppose $R/I$ is a field, but $I$ is not a maximal ideal, so there exists a proper ideal $J$ satisfying

$I \subset J \subseteq R$.

Take $j \in J$. Since $R/I$ is a field, there exists $j'$ such that

$$(j' + [I]) \cdot (j + [I]) = 1 \text{ which implies } jj' + (j + j')[I] + [I] = 1.$$

But $(j + j')[I] + [I] = 0$ in $R/I$, so $jj' = 1$ in $R/I$, hence $J$ is not a proper ideal, a contradiction.

**Exercise 2.11.** A local ring is a ring with a unique maximal ideal $m$. Prove that in a local ring, if $f \not\in m$, then $f$ is a unit.
Exercise 2.12. A ring is a Principal Ideal Domain (PID) if it is an integral domain, and every ideal is principal. In Chapter 3, we will use the Euclidean algorithm to show that \( \mathbb{K}[x] \) is a PID. Find a generator for \( \langle x^4 - 1, x^3 - 3x^2 + 3x - 1 \rangle \).

Is \( \mathbb{K}[x, y] \) a PID? \( \diamond \)

2.2. Tensor product. From the viewpoint of the additive structure, a module is just an abelian group, hence the operations of intersection, sum, and direct sum that we defined for groups can be carried out for modules \( M_1 \) and \( M_2 \) over a ring \( R \). When \( M_1 \) is a module over \( R_1 \) and \( M_2 \) is a module over \( R_2 \), a homomorphism \( R_1 \xrightarrow{\phi} R_2 \) allows us to give \( M_2 \) the structure of an \( R_1 \)-module via

\[
 r_1 \cdot m_2 = \phi(r_1) \cdot m_2.
\]

We can also use \( \phi \) to make \( M_1 \) into an \( R_2 \)-module via tensor product.

Definition 2.13. Let \( M \) and \( N \) be \( R \)-modules, and let \( P \) be the free \( R \)-module generated by \( \{(m, n) | m \in M, n \in N \} \). Let \( Q \) be the submodule of \( P \) generated by

\[
 (m_1 + m_2, n) - (m_1, n) - (m_2, n)
 (m, n_1 + n_2) - (m, n_1) - (m, n_2)
 (rm, n) - r(m, n)
 (m, rn) - r(m, n).
\]

The tensor product is the \( R \)-module:

\[
 M \otimes_R N = P/Q.
\]

We write \( m \otimes n \) to denote the class \( (m, n) \).

This seems like a strange construction, but with a bit of practice, tensor product constructions become very natural. The relations \( (rm, n) \sim r(m, n) \) and \( (m, rn) \sim r(m, n) \) show that tensor product is \( R \)-linear.

Example 2.14. For a vector space \( V \) over \( \mathbb{K} \), the tensor algebra \( T(V) \) is constructed iteratively as follows. Let \( V^i = V \otimes V \otimes \cdots \otimes V \) be the \( i \)-fold tensor product, with \( V^0 = \mathbb{K} \). Then

\[
 T(V) = \bigoplus_i V^i
\]

is the tensor algebra. The symmetric algebra \( Sym(V) \) is obtained by quotienting \( T(V) \) by the relation \( v_i \otimes v_j - v_j \otimes v_i = 0 \). When \( V = \mathbb{K}^n \), \( Sym(V) \) is isomorphic to the polynomial ring \( \mathbb{K}[x_1, \ldots, x_n] \). The exterior algebra \( \Lambda(V) \) is defined in similar fashion, except that we quotient by the relation is \( v_i \otimes v_j + v_j \otimes v_i = 0 \).
Exercise 2.15. For \( a, b \in \mathbb{Z} \), show that
\[
(Z/a\mathbb{Z}) \otimes \mathbb{Z} (Z/b\mathbb{Z}) \simeq \mathbb{Z}/\gcd(a, b)\mathbb{Z}.
\]
In particular, when \( a \) and \( b \) are relatively prime, the tensor product is zero. □

If \( M \) and \( N \) are \( R \)-modules, then a map
\[
M \times N \xrightarrow{f} P
\]
is bilinear if \( f(rm_1 + m_2, n) = rf(m_1, n) + f(m_2, n) \), and similarly in the second coordinate. Tensor product converts \( R \)-bilinear maps into \( R \)-linear maps, and possesses a universal mapping property: given a bilinear map \( f \), there is a unique \( R \)-linear map \( M \otimes_R N \rightarrow P \) making the following diagram commute:

\[
\begin{array}{ccc}
M \times N & \xrightarrow{f} & P \\
\downarrow & & \downarrow \\
M \otimes N & & \\
\end{array}
\]

Exercise 2.16. Prove the universal mapping property of tensor product. □

The motivation to define tensor product was to give the \( R_1 \)-module \( M_1 \) the structure of a \( R_2 \)-module. This operation is known as extension of scalars. The map
\[
\phi : R_1 \rightarrow R_2
\]
makes \( R_2 \) into an \( R_1 \)-module, so we can tensor \( M_1 \) and \( R_2 \) over \( R_1 \) to obtain
\[
M_1 \otimes_{R_1} R_2
\]
which is both an \( R_1 \)-module and an \( R_2 \)-module.

What is the effect tensoring a short exact sequence of \( R \)-modules
\[
0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0
\]
with an \( R \)-module \( M \)? It turns out that exactness is preserved at all but the leftmost position.

Exercise 2.17. Show that if \( M \) is an \( R \)-module and \( A_\bullet \) is a short exact sequence as above, then
\[
A_1 \otimes_R M \rightarrow A_2 \otimes_R M \rightarrow A_3 \otimes_R M \rightarrow 0.
\]
is exact. Show that if \( R = \mathbb{Z} \), then tensoring the exact sequence
\[
0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0
\]
with \( \mathbb{Z}/2 \) does not preserve exactness in the leftmost position. □
2.3. **Hom.** For a pair of $R$-modules $M_1$ and $M_2$, the set of all $R$-module homomorphisms from $M_1$ to $M_2$ is itself an $R$-module, denoted $\text{Hom}_R(M_1, M_2)$. To determine the module structure, we examine how the result acts on an element $m_1 \in M_1$, which leads to the following $R$-module structure on $\text{Hom}_R(M_1, M_2)$:

$$(\phi_1 + \phi_2)(m_1) = \phi_1(m_1) + \phi_2(m_1) \quad \text{and} \quad (r \cdot \phi_1)(m_1) = r \cdot \phi_1(m_1).$$

Given $\psi \in \text{Hom}_R(M_1, M_2)$ and $\phi \in \text{Hom}_R(M_2, N)$, we can compose them:

$$\phi \circ \psi \in \text{Hom}_R(M_1, N).$$

Put differently, we can apply $\text{Hom}_R(\bullet, N)$ to input $M_1 \xrightarrow{\phi} M_2$, yielding output $\text{Hom}_R(M_2, N) \longrightarrow \text{Hom}_R(M_1, N)$ via $\psi \mapsto \phi \circ \psi$.

When we applied $\bullet \otimes_R M$ to a short exact sequence, we preserved the direction of maps in the sequence, while losing exactness on the left. Almost the same behavior occurs if we apply $\text{Hom}_R(N, \bullet)$ to a short exact sequence, except that we lose exactness at the rightmost position. On the other hand, as we saw above, applying $\text{Hom}_R(\bullet, N)$ reverses the direction of the maps:

**Exercise 2.18.** Show that a short exact sequence of $R$-modules

$$0 \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \longrightarrow 0$$

gives rise to a left exact sequence:

$$0 \longrightarrow \text{Hom}_R(M_0, N) \longrightarrow \text{Hom}_R(M_1, N) \longrightarrow \text{Hom}_R(M_2, N)$$

Use the short exact sequence of $\mathbb{Z}$-modules in Exercise 2.17 to show that exactness can fail at the rightmost position. ♦

To represent an element $\phi \in \text{Hom}_R(M_1, M_2)$, we need to account for the fact that modules have both *generators* and *relations*.

**Definition 2.19.** A presentation for an $R$-module $M$ is a right exact sequence of the form

$$F \xrightarrow{\alpha} G \longrightarrow M \longrightarrow 0,$$

where $F$ and $G$ are free modules. If $M$ is finitely generated, then $G$ can be chosen to have finite rank, so is isomorphic to $R^a$ for $a \in \mathbb{N}$.

**Algorithm 2.20.** To define a homomorphism between finitely presented $R$-modules $M_1$ and $M_2$, take presentations for $M_1$ and $M_2$

$$R^{a_1} \xrightarrow{\alpha} R^{a_0} \longrightarrow M_1 \longrightarrow 0,$$

and

$$R^{b_1} \xrightarrow{\beta} R^{b_0} \longrightarrow M_2 \longrightarrow 0.$$
An element of $\text{Hom}_R(M_1, M_2)$ is determined by a map $R^{a_0} \xrightarrow{\gamma} R^{b_0}$ which preserves the relations. So if $b = \alpha(a)$, then $\gamma(b) = \beta(c)$. In particular, the image of the composite map

$$R^{a_1} \xrightarrow{\gamma \cdot \alpha} R^{b_0}$$

must be contained in the image of $\beta$.

**Exercise 2.21.** For $R$–modules $M, N,$ and $P$, prove that

$$\text{Hom}_R(M \otimes_R N, P) \simeq \text{Hom}_R(M, \text{Hom}_R(N, P)),$$

as follows: let

$$\phi \in \text{Hom}_R(M \otimes_R N, P).$$

Given $m \in M$, we must produce an element of $\text{Hom}_R(N, P)$. Since $\phi(m \otimes \bullet)$ takes elements of $N$ as input and returns elements of $P$ as output, it suffices to show that $\phi(m \otimes \bullet)$ is a homomorphism of $R$-modules, and in fact an isomorphism.

### 3. Localization of Rings and Modules

The process of quotienting an object $M$ by a subobject $N$ has the effect of making $N$ equal to zero. Localization simplifies a module or ring in a different way, by making some subset of objects invertible.

**Definition 3.1.** Let $R$ be a ring, and $S$ a multiplicatively closed set containing 1. Define an equivalence relation on \( \{ \frac{a}{b} \mid a \in R, b \in S \} \) via

$$\frac{a}{b} \sim \frac{c}{d} \text{ if } (ad - bc)u = 0 \text{ for some } u \in S.$$ 

Then the localization of $R$ at $S$ is

$$R_S = \left\{ \frac{a}{b} \mid a \in R, b \in S \right\} / \sim.$$ 

The object $R_S$ is a ring, with operations defined exactly as we expect:

$$\frac{a}{b} \cdot \frac{a'}{b'} = \frac{a \cdot a'}{b \cdot b'} \text{ and } \frac{a}{b} + \frac{a'}{b'} = \frac{a \cdot b' + b \cdot a'}{b \cdot b'}.$$ 

**Exercise 3.2.** Let $R = \mathbb{Z}$, and let $S$ consist of all nonzero elements of $R$. Prove that the localization $R_S = \mathbb{Q}$. More generally, prove that if $R$ is a domain and $S$ is the set of nonzero elements of $R$, then $R_S$ is the field of fractions of $R$.

The most frequently encountered situations for localization are when

- $S$ is the complement of a prime ideal.
- $S = \{1, f, f^2, \ldots \}$ for some $f \in R$.  

When $S$ is the complement of a prime ideal $P$, it is usual to write $R_P$ for $R_S$. By construction, everything outside the ideal $PR_P$ is a unit, so $R_P$ has a unique maximal ideal and is thus a local ring. If $M$ is an $R$-module, then we can construct an $R_S$-module $M_S$ in the same way $R_S$ was constructed. One reason that localization is a useful tool is that it preserves exact sequences.
Theorem 3.3. Localization preserves exact sequences.

Proof. First, suppose we have a map of \( R \)-modules \( M \xrightarrow{\phi} M' \). Since \( \phi \) is \( R \)-linear, this gives us a map \( M_S \xrightarrow{\phi_S} M'_S \) via \( \phi_S(mS) = \phi(m)s \). Let

\[
0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0
\]

be an exact sequence. Then

\[
\psi_S \phi_S \left( \frac{m}{s} \right) = \frac{\psi(\phi(m))}{s} = 0.
\]

On the other hand, if \( \frac{m}{s} \in \ker \psi_S \), so that \( \psi_S(m) = 0 \) in \( R \). But \( \psi_S(m) = \psi(sm) \) so \( sm \in \ker \psi = \text{im} \phi \), and \( sm = \phi(n) \). Thus, we have \( m = \frac{\phi(n)}{s} \) and so \( \frac{m}{s} = \frac{\phi(n)}{ss'} \).

Exercise 3.4. Let \( M \) be a finitely generated \( R \)-module, and \( S \) a multiplicatively closed set. Show that \( M_S = 0 \) iff there exists \( s \in S \) such that \( s \cdot M = 0 \).

Example 3.5. For the ideal \( I = \langle xy, xz \rangle \subseteq \mathbb{K}[x, y, z] = R \), the quotient \( R/I \) is both a ring and an \( R \)-module. It is easy to see that

\[
I = \langle x \rangle \cap \langle y, z \rangle
\]

and by Theorem 2.10, \( P = \langle x \rangle \) and \( Q = \langle y, z \rangle \) are prime ideals. What is the effect of localization? Notice that \( x \notin Q \) and \( y, z \notin P \). In the localization \( (R/I)_Q \), \( x \) is a unit, so

\[
I_Q \simeq \langle y, z \rangle_Q, \text{so } (R/I)_Q \simeq R_Q/I_Q \simeq \mathbb{K}(x),
\]

where \( \mathbb{K}(x) = \{ \frac{f(x)}{g(x)} \mid g(x) \neq 0 \} \). In similar fashion, in \( (R/I)_P \), \( y \) is a unit, so

\[
I_P \simeq \langle x \rangle_P, \text{so } (R/I)_P \simeq \mathbb{K}(y, z).
\]

Finally, if \( T \) is a prime ideal which does not contain \( I \), then there is an element \( f \in I \setminus T \). But then

- \( f \) is a unit because it is outside \( T \)
- \( f \) is zero because it is inside \( I \).

Hence

\( (R/I)_T = 0 \) if \( I \nsubseteq T \).

Exercise 3.6. Carry out the same computation for \( I = \langle x^2, xy \rangle \subseteq \mathbb{K}[x, y] = R \).

You may find it useful to use the fact that

\[
I = \langle x^2, y \rangle \cap \langle x \rangle
\]

Hint: \( \langle x^2, y \rangle \) is not a prime ideal, but \( \langle x, y \rangle \) is prime.
4. Noetherian rings, Hilbert basis theorem, Varieties

Exercise 2.12 defined a principal ideal domain; the ring \( \mathbb{K}[x] \) of polynomials in one variable with coefficients in a field is an example. In particular, every ideal \( I \subseteq \mathbb{K}[x] \) can be generated by a single element, hence the question of when \( f(x) \in I \) is easy to solve: find the generator \( g(x) \) for \( I \), and check if \( g(x) | f(x) \). While this is easy in the univariate case, in general the ideal membership problem is difficult. The class of Noetherian rings includes all principal ideal domains, but is much larger; in a Noetherian ring every ideal is finitely generated. Gröbner bases and the Buchberger algorithm are analogs of Gaussian Elimination for polynomials of degree larger than one, and provide a computational approach to tackle the ideal membership question. For details on this, see [33].

4.1. Noetherian Rings.

Definition 4.1. A ring is Noetherian if it contains no infinite ascending chains of ideals: there is no infinite chain of proper inclusions of ideals as below

\[ I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots \]

A module is Noetherian if it contains no infinite ascending chains of submodules. A ring is Noetherian exactly when all ideals finitely generated.

Theorem 4.2. A ring \( R \) is Noetherian iff every ideal is finitely generated.

Proof. Suppose every ideal in \( R \) is finitely generated, but there is an infinite ascending chain of ideals:

\[ I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots \]

Let \( J = \bigcup_{i=1}^\infty I_i \). Since \( j_1 \in J, j_2 \in J \) and \( r \in R \) implies \( j_1 + j_2 \in J \) and \( r \cdot j_i \in J \), \( J \) is an ideal. By assumption, \( J \) is finitely generated, say by \( \{f_1, \ldots, f_k\} \), and each \( f_i \in I_{l_i} \) for some \( l_i \). So if \( m = \max\{l_i\} \) is the largest index, we have

\[ I_{m-1} \subsetneq I_m = I_{m+1} = \cdots, \]

a contradiction. Now suppose that \( I \) cannot be finitely generated, so we can find a sequence of elements \( \{f_1, f_2, \ldots\} \) of \( I \) with \( f_i \notin \langle f_1, f_2, \ldots, f_{i-1} \rangle \). This yields

\[ \langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \langle f_1, f_2, f_3 \rangle \subsetneq \cdots, \]

which is an infinite ascending chain of ideals. \( \square \)

Exercise 4.3. Let \( M \) be a module. Prove the following are equivalent:

(a) \( M \) contains no infinite ascending chains of submodules.

(b) Every submodule of \( M \) is finitely generated.

(c) Every nonempty subset \( \Sigma \) of submodules of \( M \) has a maximal element with respect to inclusion.

The last condition says that \( \Sigma \) is a special type of partially ordered set. \( \diamond \)
Exercise 4.4. Prove that if $R$ is Noetherian and $M$ is a finitely generated $R$-module, then $M$ is Noetherian, as follows. Since $M$ is finitely generated, there exists an $n$ such that $R^n$ surjects onto $M$. Suppose there is an infinite ascending chain of submodules of $M$, and consider what this would imply for $R$. $\diamond$

Theorem 4.5. [Hilbert Basis Theorem] If $R$ is a Noetherian ring, then so is $R[x]$.

Proof. Let $I$ be an ideal in $R[x]$. By Theorem 4.2 we must show that $I$ is finitely generated. The set of lead coefficients of polynomials in $I$ generates an ideal $I'$ of $R$, which is finitely generated, because $R$ is Noetherian. Let $I' = \langle g_1, \ldots, g_k \rangle$. For each $g_i$ there is a polynomial

$$f_i \in I, f_i = g_ix^{m_i} + \text{terms of lower degree in } x.$$ 

Let $m = \max \{m_i\}$, and let $I''$ be the ideal generated by the $f_i$. Given any $f \in I$, reduce it modulo members of $I''$ until the lead term has degree less than $m$. The $R$-module $M$ generated by $\{1, x, \ldots, x^{m-1}\}$ is finitely generated, hence Noetherian. Therefore the submodule $M \cap I$ is also Noetherian, with generators $\{h_1, \ldots, h_j\}$. Hence $I$ is generated by $\{h_1, \ldots, h_j, g_1, \ldots, g_k\}$, which is a finite set. $\Box$

When a ring $R$ is Noetherian, even for an ideal $I \subseteq R$ specified by an infinite set of generators, there will exist a finite generating set for $I$. A field $K$ is Noetherian, so the Hilbert Basis Theorem and induction tell us that the ring $K[x_1, \ldots, x_n]$ is Noetherian, as is a polynomial ring over $\mathbb{Z}$ or any other principal ideal domain. Thus, the goal of determining a finite generating set for an ideal is attainable.

4.2. Solutions to a polynomial system: Varieties. In linear algebra, the objects of study are the solutions to systems of polynomial equations of the simplest type: all polynomials are of degree one. Algebraic geometry is the study of the sets of solutions to systems of polynomial equations of higher degree. The choice of field is important: $x^2 + 1 = 0$ has no solutions in $\mathbb{R}$ and two solutions in $\mathbb{C}$; in linear algebra this issue arose when computing eigenvalues of a matrix.

Definition 4.6. A field $K$ is algebraically closed if every nonconstant polynomial $f(x) \in K[x]$ has a solution $f(p) = 0$ with $p \in K$. The algebraic closure $\overline{K}$ is the smallest field containing $K$ which is algebraically closed.

Given a system of polynomial equations $\{f_1, \ldots, f_k\} \subseteq R = K[x_1, \ldots, x_n]$, note that the set of solutions over $K$ depends only on the ideal

$$I = \langle f_1, \ldots, f_k \rangle = \{\sum_{i=1}^k g_if_i \mid g_i \in R\}$$

We denote the solution set as

$$\mathbf{V}(I) \subseteq K^n \subseteq \mathbb{K}^n$$
Adding more equations to a polynomial system imposes additional constraints on the solutions, hence passing to varieties reverses inclusion

\[ I \subseteq J \Rightarrow V(J) \subseteq V(I) \]

Since ideals \( I, J \subseteq R \) are submodules of the same ambient module, we have

- \( I \cap J = \{ f \mid f \in I \text{ and } f \in J \} \).
- \( IJ = \{ fg \mid f \in I \text{ and } g \in J \} \).
- \( I + J = \{ f + g \mid f \in I \text{ and } g \in J \} \).

It is easy to check these are all ideals.

**Exercise 4.7.** Prove that

\[ V(I \cap J) = V(I) \cup V(J) = V(IJ) \]

and that \( V(I + J) = V(I) \cap V(J) \). \( \diamond \)

**Definition 4.8.** The radical of an ideal \( I \subseteq R \) is

\[ \sqrt{I} = \{ f \mid f^m \in I \text{ for some power } m \} \].

**Exercise 4.9.** Show if \( I = \langle x^2 - xz, xy - yz, xz - z^2, xy - xz, y^2 - yz, yz - z^2 \rangle \) in \( \mathbb{K}[x, y, z] \), then \( \sqrt{I} = \langle x - z, y - z \rangle \). \( \diamond \)

**Definition 4.10.** For ideals \( I, J \subseteq R \), the ideal quotient (or colon ideal) is

\[ I : J = \{ f \mid f \cdot J \subseteq I \} \].

**Exercise 4.11.** Let \( I \) and \( J = \sqrt{I} \) be as in Exercise 4.9. Show \( I : J = \langle x, y, z \rangle \). \( \diamond \)

The operations of radical and ideal quotient have geometric interpretations; we tackle the radical first. Given \( X \subseteq \mathbb{K}^n \), consider the set \( I(X) \) of all polynomials in \( R = \mathbb{K}[x_1, \ldots, x_n] \) which vanish on \( X \).

**Exercise 4.12.** Prove that \( I(X) \) is an ideal, and in fact a radical ideal. Next, prove that \( V(I(X)) \) is the smallest variety containing \( X \). \( \diamond \)

Notice that for \( I = \langle x^2 + 1 \rangle \subseteq \mathbb{R}[x] \subseteq \mathbb{C}[x] \), \( V(I) \) is empty in \( \mathbb{R} \), but consists of two points in \( \mathbb{C} \). This brings us Hilbert’s *Nullstellensatz* (see [47] for a proof).

**Theorem 4.13.** If \( \mathbb{K} \) is algebraically closed, then

- Hilbert *Nullstellensatz* (Weak version): \( V(I) = \emptyset \iff 1 \in I \).
- Hilbert *Nullstellensatz* (Strong version): \( I(V(I)) = \sqrt{I} \).

An equivalent formulation is that over an algebraically closed field, there is a 1:1 correspondence between maximal ideals \( I(p) \) and points \( p \). For an arbitrary set \( X \subseteq \mathbb{K}^n \), the smallest variety containing \( X \) is \( V(I(X)) \). It is possible to define a topology, called the Zariski topology on \( \mathbb{K}^n \), where the closed sets are of the form \( V(I) \), and when working with this topology we write \( \mathbb{A}^n_{\mathbb{K}} \) and speak of affine space.
While it is not needed for these notes, we touch on the Zariski topology briefly in the general discussion of topological spaces in Chapter 4. We close with the geometry of the ideal quotient operation. Suppose we know there is some spurious or unwanted component \( V(J) \subseteq V(I) \). How do we remove \( V(J) \)? Equivalently, what is the smallest variety containing \( V(I) \setminus V(J) \)?

**Theorem 4.14.** The variety \( V(I(\mathbb{V}(I) \setminus V(J))) \subseteq V(I : J) \).

**Proof.** Since \( I_1 \subseteq I_2 \Rightarrow V(I_2) \subseteq V(I_1) \), it suffices to show

\[
I : J \subseteq I(\mathbb{V}(I) \setminus \mathbb{V}(J)).
\]

If \( f \in I : J \) and \( p \in \mathbb{V}(I) \setminus \mathbb{V}(J) \), then since \( p \notin V(J) \) there is a \( g \in J \) with \( g(p) \neq 0 \). Since \( f \in I : J \), \( fg \in I \), and so

\[
p \in V(I) \Rightarrow f(p)g(p) = 0.
\]

As \( g(p) \neq 0 \), this forces \( f(p) = 0 \) and therefore \( f \in I(\mathbb{V}(I) \setminus \mathbb{V}(J)) \).

\[ \square \]

**Example 4.15.** Projective space \( \mathbb{P}^n_K \) over a field \( K \) is defined as 

\[
\mathbb{P}^n_K = \mathbb{K}^{n+1} \setminus \{0\} / \sim \text{ where } p_1 \sim p_2 \text{ iff } p_1 = \lambda p_2 \text{ for some } \lambda \in \mathbb{K}^*.
\]

One way to visualize \( \mathbb{P}^n_K \) is as the set of lines thru the origin in \( \mathbb{K}^{n+1} \).

Writing \( V(x_0) \) for the points where \( x_0 \neq 0 \), we have that \( \mathbb{K}^n \simeq V(x_0) \subseteq \mathbb{P}^n_K \) via \((a_1, \ldots, a_n) \mapsto (1, a_1, \ldots, a_n)\). This can be quite useful, since \( \mathbb{P}^n_K \) is compact.

**Exercise 4.16.** Show that a polynomial \( f = \sum c_{\alpha_i} x^{\alpha_i} \in \mathbb{K}[x_0, \ldots, x_n] \) has a well-defined zero set in \( \mathbb{P}^n_K \) iff it is homogeneous: the exponents all have the same weight. Put differently, all the \( \alpha_i \) have the same dot product with the vector \([1, \ldots, 1]\). Show that a homogeneous polynomial \( f \) does not define a function on \( \mathbb{P}^n_K \), but that the rational function \( \frac{f}{g} \) with \( f \) and \( g \) homogeneous of the same degree does define a function on \( \mathbb{P}^n_K \setminus V(g) \).

\[ \diamond \]

**Example 4.17.** Let \( I \) be the ideal of \( 2 \times 2 \) minors of

\[
\begin{bmatrix}
x & y & z \\
y & z & w
\end{bmatrix}, \text{ so } I = \langle xz - y^2, xw - yz, yw - z^2 \rangle,
\]

and let \( J = \langle xz - y^2, xw - yz \rangle \) and \( L = \langle x, y \rangle \). Then \( V(I) \) is a curve in \( \mathbb{P}^3 \):

\[ V(J) = V(I) \cup V(L) \]

**Exercise 4.18.** Prove the equality above. Hint: use ideal quotient.  

\[ \diamond \]
A fundamental result in topological data analysis is that persistent homology can be represented as a barcode; short bars in the barcode represent homology classes that are transient whereas long bars represent persistent features.

The intuition is that noise in the data corresponds to features that have short bars; long bars represent underlying structure in the data. The result follows from a central structure theorem in algebra: the decomposition theorem for modules over a principal ideal domain. This chapter focuses on the structure theorem; one surprise is that the theorem has applications ranging from the theory of finitely generated Abelian groups to the Jordan block decomposition of matrices.

**Theorem 0.1.** A finitely generated module $M$ over a principal ideal domain (PID) $R$ has a direct sum decomposition as defined in Example 2.3 of Chapter 2:

$$M \cong R^n \bigoplus_{i=1}^{m} R/\langle d_i \rangle \text{ with } d_i | d_{i+1} \text{ and } 0 \neq d_i \text{ a nonunit}.$$  

Since $R$ is a principal ideal domain, $d_i | d_{i+1}$ is equivalent to $d_{i+1} \in \langle d_i \rangle$. The term $R^n$ is the free component of $M$, and the term $\bigoplus R/\langle d_i \rangle$ is the torsion component.
If $R$ is a domain, an $R$-module $M$ is torsion if $\text{ann}(m) \neq 0$ for all $m \in M$; $\oplus R/\langle d_i \rangle$ is a torsion module. In the context of persistent homology, the free component corresponds to long bars, and the torsion component corresponds to short bars. In linear algebra, a consequence of Theorem 0.1 is that a linear operator
\[ V \xrightarrow{T} V \]
on a finite dimensional vector space $V$ can be written after a change of basis as a block diagonal matrix, where the blocks along the diagonal are of the form
\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & a_1 \\
1 & \ddots & \cdots & 0 & a_2 \\
0 & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & a_m
\end{bmatrix}
\]
Such a decomposition is the Rational Canonical Form of the matrix. As we saw in the first chapter, not every square matrix can be diagonalized; the rational canonical form is the closest we can get, in general, to diagonalization. In this chapter, we will cover the following topics:
- Principal Ideal Domains and Euclidean Domains.
- Rational Canonical Form of a Matrix.
- Linear Transformations, $\mathbb{K}[t]$-Modules, Jordan Form.
- Structure of Abelian Groups and Persistent Homology.

1. Principal Ideal Domains and Euclidean Domains

Exercise 2.12 of Chapter 2 defined a principal ideal domain: a ring which is an integral domain ($a \cdot b = 0$ implies either $a$ or $b$ is zero), and where every ideal is principal (so can be generated by a single element). In §4 we prove Theorem 0.1 with the additional hypothesis that the PID is a Euclidean Domain.

**Definition 1.1.** An integral domain $R$ is a Euclidean Domain if it possesses a Euclidean norm, which is a function
\[ R \setminus 0 \xrightarrow{v} \mathbb{Z}_{\geq 0} \text{ such that} \]
- For all $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that
  \[ a = bq + r, \text{ with } r = 0 \text{ or } v(r) < v(a). \]
- For all $a \neq 0 \neq b \in R$, $v(a) \leq v(a \cdot b)$
**Theorem 1.2.** A Euclidean Domain $R$ is a Principal Ideal Domain.

**Proof.** Let $I \subseteq R$ be an ideal; if $I = \langle 0 \rangle$ there is nothing to prove, so suppose $I$ is nonzero. Choose $f \in I$ with $v(f)$ minimal. Any $g \in I$ satisfies

$$g = f \cdot q + r,$$

with $r = 0$ or $v(r) < v(f)$. Since

$$r = (g - f \cdot q) \in I$$

and $f$ was chosen with $v(f)$ minimal, $v(r)$ cannot be less than $v(f)$, so $r = 0$, hence any $g$ is a multiple of $f$ and $I = \langle f \rangle$. □

**Example 1.3.** Examples of Euclidean norms

- $R = \mathbb{Z}$, with norm $v(a) = |a|$.
- $R = \mathbb{K}[x]$, with norm $v(f(x)) = \text{degree}(f(x))$.

**Exercise 1.4.** Prove the two norms above are Euclidean norms, so $\mathbb{Z}$ and $\mathbb{K}[x]$ are Euclidean Domains, and hence by Theorem 1.2 also Principal Ideal Domains. □

**Algorithm 1.5.** The Euclidean Algorithm. In a Euclidean domain, the Euclidean algorithm allows us to find the greatest common divisor (GCD) of two nonzero elements: the largest $c$ such that $c \mid a$ and $c \mid b$. To compute $\text{GCD}(a, b)$

(a) Form $a = bq_1 + r_1$, where $r_1 = 0$ or $v(r_1) < v(b)$.

(b) If $r_1 = 0$ then $b \mid a$, done. Else since $v(r_1) < v(b)$, $b = r_1q_2 + r_2$.

(c) Iterate until

(d) $r_{i-1} = r_iq_{i+1} + r_{i+1}$ with $r_{i+1} = 0$

The algorithm terminates, since the $r_i$ are a decreasing sequence in $\mathbb{Z}_{\geq 0}$, and

$$\text{GCD}(a, b) = \text{the last nonzero } r_i$$

which follows by reversing the process.

**Example 1.6.** To compute $\text{GCD}(48, 332)$, we proceed as follows:

(a) $332 = 48 \cdot 6 + 44$.

(b) $48 = 44 \cdot 1 + 4$

(c) $44 = 4 \cdot 11 + 0$

So $\text{GCD}(48, 332) = 4$. To check, note that $48 = 4 \cdot 12$ and $332 = 4 \cdot 83$.

**Exercise 1.7.** GCD computations.

(a) Show $\text{GCD}(4, 83) = 1$.
(b) Show $\text{GCD}(462, 90) = 6$.
(c) Show $\text{GCD}(105, 126) = 21$.

It is nontrivial to construct PID’s which are not Euclidean Domains. □
2. Rational Canonical Form of a Matrix

As noted in the introduction to this chapter, the Rational Canonical Form (RCF) of a square matrix is, in general, the closest we can get to diagonalization for an arbitrary square matrix.

**Definition 2.1.** An invariant block is an $m \times m$ matrix of the form of Equation 0.1: zero except in positions $a_{i,i-1} = 1$ and $a_{i,m}$.

**Theorem 2.2.** Any $n \times n$ matrix is similar to a block diagonal matrix

\[
\begin{pmatrix}
B_{i_1} & 0 & \cdots & \cdots & 0 \\
0 & B_{i_2} & \ddots & \cdots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
0 & 0 & \ddots & 0 & \vdots \\
0 & \cdots & \cdots & 0 & B_{i_k}
\end{pmatrix}
\]

(2.1)

with $B_{i_j}$ an invariant block, and

\[\sum_{j=1}^{k} i_j = n.\]

**Proof.** An $n \times n$ matrix represents, with respect to some basis, a linear transformation $T : V \rightarrow V$. Take an element $v \in V$, and let $j$ be the smallest integer such that

$T^j(v) \in \text{Span}(B)$, with $B = \{v, T(v), T^2(v), \cdots, T^{j-1}(v)\}$.

If we write $V_{j-1}$ for $\text{Span}(B)$, then $B$ is a basis for $V_{j-1}$: by construction it is a spanning set, and linearly independent by the choice of $j$ as minimal. Note that $T$ restricts to a linear transformation on $V_{j-1}$. Write

\[T^j(v) = \sum_{i=0}^{j-1} a_i T^i(v).\]

As $v \mapsto T(v), T(v) \mapsto T^2(v)$ and so on, with respect to the ordered basis $B$, the matrix for $T$ restricted to $V_{j-1}$ is

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & a_0 \\
1 & \ddots & \cdots & 0 & 1 \\
0 & 1 & \ddots & 0 & \vdots \\
\vdots & 0 & \ddots & 0 & \vdots \\
0 & \cdots & 0 & 1 & a_{j-1}
\end{pmatrix}
\]

(2.2)

Now iterate the process for $V/V_{j-1}$. □

**Exercise 2.3.** Find the RCF of the $4 \times 4$ matrix in Chapter 1, Example 4.7. ♦
3. Linear Transformations, $\mathbb{K}[t]$-Modules, Jordan Form

In this section, we connect the rational canonical form of a matrix to finitely generated modules over the ring $\mathbb{K}[t]$.

**Theorem 3.1.** There is a one to one correspondence between the set of linear transformations $V \rightarrow V$ of a finite dimensional vector space $V$ over $\mathbb{K}$, and the set of finitely generated torsion $\mathbb{K}[t]$-modules.

**Proof.** First, let

$$V \xrightarrow{T} V$$

be a linear transformation, and $f(t) \in \mathbb{K}[t]$ be given by

$$f(t) = \sum_{i=0}^{m} a_{i}t^{i}.$$  

Define an action of $f(t)$ on $v \in V$ via

$$f(t) \cdot v = \sum_{i=0}^{m} a_{i}T^{i}(v).$$

A check shows that this satisfies the module axioms of Chapter 2, Definition 2.1. Now suppose $M$ is a finitely generated torsion $\mathbb{K}[t]$-module, so the constants $\mathbb{K}$ act on $M$, and the module properties of this action are identical to the vector space properties, so $M$ is a vector space over $\mathbb{K}$. To determine the linear transformation $T$, consider the action of multiplication by $t$. The module properties imply that

$$t \cdot (cv + w) = ct \cdot v + t \cdot w \text{ for } c \in \mathbb{K} \text{ and } v, w \in M.$$  

Defining $T = t$ yields a linear transformation. \qed

**Example 3.2.** Suppose $V = \mathbb{K}[t]/\langle f(t) \rangle$, where $f(t) = t^{j} + a_{j-1}t^{j-1} + \cdots + a_{0}$. Then with respect to the basis

$$\{1, t, t^{2}, \ldots, t^{j-1}\}$$

the linear transformation $T = t$ is represented by the matrix of Equation 2.2, with $a_{i}$ replaced by $-a_{i}$.

**Example 3.3.** For an algebraically closed field such as $\mathbb{K} = \mathbb{C}$, every polynomial in $\mathbb{K}[t]$ factors as a product of linear forms. The *Jordan Canonical Form* of a matrix exploits this fact; albeit with a slightly different basis. First, let

$$f(t) = (t - \alpha)^{n} \text{ and } W = \mathbb{K}[t]/\langle f(t) \rangle.$$  

Let $w_{0}$ denote the class of 1 in $W$, and $w_{i} = (t - \alpha)^{i} \cdot w_{0}$. Then

$$(t - \alpha) \cdot w_{0} = w_{1}$$

$$\vdots$$

$$(t - \alpha) \cdot w_{n-2} = w_{n-1}$$

$$(t - \alpha) \cdot w_{n-1} = 0$$
Letting $T = \cdot t$, we have that

$$T(w_i) = \alpha w_i + w_{i+1} \text{ for } i \in \{0, \ldots, n-2\}$$

and $T(w_{n-1}) = \alpha w_{n-1}$. In this basis, the matrix for $T$ is a Jordan block

$$T = \begin{bmatrix}
\alpha & 0 & \cdots & \cdots & 0 \\
1 & \alpha & 0 & \cdots & \vdots \\
0 & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \alpha & 0 \\
0 & \cdots & 0 & 1 & \alpha 
\end{bmatrix}.$$ 

In general, over an algebraically closed field we have

$$f(t) = \prod_{j=1}^{k} (t - \alpha_j)^{m_j}$$

and in the representation of Theorem 2.2 the blocks $B_{i\ell}$ are replaced with Jordan blocks with $\alpha_j$ on the diagonal.

### 4. Structure of Abelian Groups and Persistent Homology

Our second application, and the one which is of key importance in topological data analysis, comes from finitely generated Abelian groups. First, note that an Abelian group $A$ is a $\mathbb{Z}$-module, because for $n \in \mathbb{Z}_{>0}$ and $a \in A$,

$$n \cdot a = a + a + \cdots + a \text{ n times.}$$

In this case, Theorem 0.1 may be stated in the form

**Theorem 4.1.** A finitely generated Abelian group has a direct sum decomposition

$$A \simeq \mathbb{Z}^n \bigoplus \mathbb{Z}/(p_i^{a_i})$$

with $p_i$ prime.

The proof of Theorem 4.1 works for any Euclidean Domain, for our data analysis application of persistent homology we will replace $\mathbb{Z}$ with $K[x]$.

**Theorem 4.2.** For an $m \times n$ matrix with entries in $\mathbb{Z}$, there exist integer matrices $B$ and $C$ which encode elementary row and column operations over $\mathbb{Z}$, such that

$$A' = BAC^{-1} = \begin{bmatrix}
d_1 & 0 & \cdots & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
\vdots & \ddots & \ddots & d_k & 0 \\
0 & 0 & \ddots & 0 & \vdots \\
0 & \cdots & \ddots & \cdots & 0
\end{bmatrix}$$

with $d_i > 0$ and $d_i|d_{i+1}$. 


Proof. The proof has two steps, and moves back and forth between them:

(a) We first use elementary row and column swaps to put the smallest entry of \( A \) in position \((1, 1)\). Next, use elementary row and column operations to zero out all entries in the first row and column, except at position \((1, 1)\). When the entries of \( A \) are in a field \( \mathbb{K} \), this is straightforward, but over \( \mathbb{Z} \) care must be taken. For example, if the first row starts as \([2, 3, \cdots]\), we can use 2 to reduce 3, with a remainder of 1. Now 1 is the smallest entry, so swap it into position \((1, 1)\), and continue.

The key point is that each entry in the first row may be written as

\[ a_{1i} = a_{11}q + r, \text{ with } r = 0 \text{ or } r < a_{11}. \]

If \( r = 0 \) then \( a_{1i} \) reduces to zero mod \( a_{11} \). Otherwise after reducing \( a_{1i} = r \), swap the first column and column \( i \) to get the smaller value \( r \) in position \((1, 1)\). This process concludes with a matrix \( A' \) which is nonzero in the first row and first column only in position \((1, 1)\), with \( A'_{1,1} = d_1 \). To achieve \( d_i | d_{i+1} \), we need to do more work.

(b) Suppose \( A' \) has an entry \( b \) which is not divisible by \( d_1 \). Add the column of \( A' \) containing \( b \) to the first column of \( A' \), and call the resulting matrix \( A'' \). Because the column of \( A' \) containing \( b \) has zero entry in the first row, and the first column of \( A' \) is zero except in position \((1, 1)\), the first column of \( A'' \) has \( b \) as an entry. Since \( a_{11} = d_1 \) does not divide \( b \), we have

\[ b = a_{11}q + r \text{ with } r \neq 0, \text{ so } r < a_{11}. \]

Swap the row with \( r \) with the first row, and return to step (a).

The process terminates: at every step, the numbers decrease in absolute value. \( \square \)

Exercise 4.3. Working with \( \mathbb{Z}/p \).

(a) Show that

\[ \mathbb{Z}/2 \times \mathbb{Z}/2 \not\cong \mathbb{Z}/4. \]

(b) Show that if \( p_1 \) and \( p_2 \) are distinct prime numbers, then

\[ \mathbb{Z}/p_1 \times \mathbb{Z}/p_2 \cong \mathbb{Z}/p_1 p_2. \]

Now show if \( R = \mathbb{Z} \), Theorem 4.1 is equivalent to Theorem 4.2. \( \diamond \)

Lemma 4.4. If \( R \) is a Noetherian ring and \( M \) a finitely generated \( R \)-module, then the module of relations \( F \) in Definition 2.19 of Chapter 2 is of finite rank.

Proof. Exercise 4.4 of Chapter 2 shows that a finitely generated module over a Noetherian ring is Noetherian, hence so is every submodule. \( \square \)

With these preliminaries in hand, we’re ready for the proof of Theorem 4.1.
Proof [of Theorem 4.1] If $A$ is a finitely generated Abelian group, then $A$ is a finitely generated $\mathbb{Z}$-module. As a principal ideal domain is Noetherian, Lemma 4.4 implies that $A$ has a presentation of the form

$$\mathbb{Z}^m \xrightarrow{\phi} \mathbb{Z}^n \rightarrow A \rightarrow 0.$$  

By Theorem 4.2,

$$\phi = \begin{bmatrix}
    d_1 & 0 & \ldots & \ldots & \ldots & 0 \\
    0 & d_2 & 0 & \ldots & \ldots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & 0 & \ldots & \ddots & \ddots & 0 \\
    0 & 0 & \ldots & \ddots & \ddots & \vdots \\
    0 & \ldots & \ddots & \ddots & \ddots & 0
\end{bmatrix} \text{ with } d_i > 0 \text{ and } d_i | d_{i+1}.$$  

Since $A = \text{coker}(\phi)$, this means exactly that

$$A \simeq \bigoplus_{i=1}^{k} \mathbb{Z}/d_i \mathbb{Z} \bigoplus \mathbb{Z}^{n-k}.$$  

For a Euclidean domain, Theorem 4.1 and Theorem 0.1 are equivalent, with small modifications this proof also works for a PID. □

Exercise 4.5. Row reduce the matrix

$$\begin{bmatrix}
    3 & 1 & -4 \\
    2 & -3 & 1 \\
    -4 & 6 & -2
\end{bmatrix}$$  

using integer elementary operations. □

In Chapter 5 we define homology, which provides a method to attach algebraic invariants to a topological space. Chapter 7 introduces persistent homology, where the algebraic invariants that are constructed are modules over $\mathbb{K}[x]$. Since $\mathbb{K}[x]$ is a Euclidean Domain, we can apply Theorem 4.1 to analyze the structure of the resulting modules. In fact, $\mathbb{K}[x]$ has additional structure as a $\mathbb{Z}$-graded ring, discussed in detail in §2 of Chapter 8. Being $\mathbb{Z}$-graded means that as a $\mathbb{K}$-vector space

$$\mathbb{K}[x] = \bigoplus_{i \in \mathbb{Z}} x^i, \text{ and that } x^i \cdot x^j = x^{i+j}.$$  

A graded module $M$ over the graded ring $\mathbb{K}[x]$ must satisfy a similar condition

$$M = \bigoplus_{i \in \mathbb{Z}} M_i, \text{ and if } m \in M_i \text{ then } x^j \cdot m \in M_{i+j}$$  

This concept of grading seems like an unnecessary accounting technicality. However, it turns out to be a main ingredient in Topological Data Analysis.
Example 4.6. We return to the figure at the beginning of this chapter. As noted
in the introduction, the free component in the decomposition is graphically repre-
sented by long bars in the barcode, and the torsion component is represented by
short bars. The graded condition means that a free summand will correspond to
a \textit{graded} ideal in \( \mathbb{K}[x] \). Since we know that \( \mathbb{K}[x] \) is a principal ideal domain, any
ideal in \( \mathbb{K}[x] \) has the form \( \langle f(x) \rangle \).

In Exercise 4.7 below, you’ll show that the graded condition means that \( f(x) \)
must be a monomial, hence of the form \( x^i \). The \textit{birth} of a longbar at time \( i \) cor-
responds exactly to \( \langle x^i \rangle \). Similarly, a graded torsion summand corresponds to
a graded ideal \( \langle x^j \rangle \). Continuing to think of the exponent as representing time,
\( \langle x^j \rangle \) represents a birth at time \( j \); because the class is torsion it \textit{dies} at a later time \( \langle x^{j+k} \rangle \).

To summarize, we have

\begin{align*}
\text{long bar} & \iff \langle x^{b_i} \rangle \\
\text{short bar} & \iff \langle x^{b_i} \rangle / \langle x^{b_i+d_i} \rangle
\end{align*}

The corresponding modules are

\begin{align*}
H_0 & = R \bigoplus_{i=1}^{13} \frac{R}{\langle x^{b_i-d_i} \rangle}. \\
H_1 & = \bigoplus_{j=1}^{16} \langle x^{b_j} \rangle / \langle x^{b_j+d_j} \rangle. \\
H_2 & = \langle x^{b_j} \rangle / \langle x^{b_j+d_j} \rangle.
\end{align*}

In particular, note that for \( H_0 \) all classes are born at time zero, so the generators
correspond to \( \langle x^0 \rangle = R \). Similarly, for \( H_1 \) and \( H_2 \), all classes are torsion, so will
have both a birth time \( b_j \) and a death time \( b_j + d_j \).

\textbf{Exercise 4.7.} Prove that a proper nonzero \( \mathbb{Z}_r \)-graded ideal in \( \mathbb{K}[x] \) must be of the
form \( \langle x^i \rangle \) for some \( i \in \mathbb{Z}_{>0} \).

\textbf{Exercise 4.8.} For an \( R \)-module \( M \), the annihilator is defined as

\( \text{Ann}(M) = \{ r \in R \mid r \cdot m = 0 \text{ for all } m \in M \} \).

Show \( \text{Ann}(M) \) is an ideal. What is the annihilator of \( \bigoplus_{j=1}^{k} \langle x^{b_j} \rangle / \langle x^{b_j+d_j} \rangle \)?
Basics of Topology: Spaces and Sheaves

In Chapter 1 we compressed a semester of linear algebra into twenty pages. Now we up our game to more advanced topics: this chapter covers a semester of topology and sheaf theory. We start in §1 with a quick review of topological basics. In §2 we discuss vector bundles, which provide a good intuition for sheaf theory, which is introduced in §3. Sheaf theory can be a somewhat daunting topic when first encountered; the main point to keep in mind is that a sheaf is really nothing more than a bookkeeping device.

- Sheaves organize local data, assigning to every open set $U$ an algebraic object $\mathcal{F}(U)$. When $V \subseteq U$, there is a restriction map

$$\mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$$

satisfying several natural conditions which appear in Definition 3.1.

- Sheaves facilitate the construction of global structures from local structures: this is often referred to as gluing, formalized in Definition 3.2

Having developed dexterity with sheaf constructions, in §4 we apply this to algebraic varieties. We also introduce a new algebraic tool: primary decomposition, which will be useful in probing the underlying geometry of sheaves which arise in Chapters 7 and 8 when we tackle topological data analysis. The topics covered are:

- Topological Spaces.
- Vector Bundles.
- Sheaf Theory.
- Sheaves on Algebraic Varieties.
1. Topological Spaces

1.1. Set theory and equivalence relations. In this section we fix notation, and review basic topology and set theory. A set $S$ is a (not necessarily finite) collection of objects $\{s_1, \ldots, s_k\}$; standard operations among sets include

- (a) $S_1 \cup S_2 = \{m \mid m \in S_1 \text{ or } m \in S_2\}$.
- (b) $S_1 \cap S_2 = \{m \mid m \in S_1 \text{ and } m \in S_2\}$.
- (c) $S_1 \setminus S_2 = \{m \mid m \in S_1 \text{ and } m \notin S_2\}$.

The Cartesian product $S_1 \times S_2$ of two sets is

$S_1 \times S_2 = \{(s_1, s_2) \mid s_i \in S_i\}$

and a relation $\sim$ between $S_1$ and $S_2$ is a subset of pairs of $S_1 \times S_2$. We say that a relation on $S \times S$ is

- (a) reflexive if $a \sim a$ for all $a \in S$.
- (b) symmetric if $a \sim b \Rightarrow b \sim a$ for all $a, b \in S$.
- (c) transitive if $a \sim b$ and $b \sim c \Rightarrow a \sim c$ for all $a, b, c \in S$.

A relation which is reflexive, symmetric, and transitive is an equivalence relation.

Exercise 1.1. Show that an equivalence relation partitions a set into disjoint subsets. Elements of a subset in which all elements are related by $\sim$ are called equivalence classes.

A relation $R \subseteq S \times S$ is a partial order on $S$ if it is transitive, and $(s_1, s_2)$ and $(s_2, s_1) \in R$ imply that $s_1 = s_2$. In this case $S$ is called a poset, which is shorthand for “partially ordered set”.

1.2. Definition of a topology. The usual setting for calculus and elementary analysis is over the real numbers; $\mathbb{R}^n$ is a metric space, with $d(p, q)$ the distance between points $p$ and $q$. A set $U \subseteq \mathbb{R}^n$ is open when for every $p \in U$ there exists $\delta_p > 0$ such that

$N_{\delta_p}(p) = \{q \mid d(q, p) < \delta_p\} \subseteq U$,

and closed when the complement $\mathbb{R}^n \setminus U$ is open. The key properties of open sets are that any (including infinite) union of open sets is open, and a finite intersection of open sets is open. Topology takes this as a cue.

Definition 1.2. A topology is a set $S$ and collection $\mathcal{U}$ of subsets $U \subseteq S$, such that

- (a) $\emptyset$ and $S$ are elements of $\mathcal{U}$.
- (b) $\mathcal{U}$ is closed under arbitrary union: for $U_i \in \mathcal{U}$, $\bigcup_{i \in I} U_i \in \mathcal{U}$.
- (c) $\mathcal{U}$ is closed under finite intersection: for $U_i \in \mathcal{U}$, $\bigcap_{i=1}^{m} U_i \in \mathcal{U}$.

Elements of $\mathcal{U}$ are the open sets of the topology.
Exercise 1.3. Prove that a topology can also be defined by specifying the closed sets, by complementing the conditions of Definition 1.2.

Since $\mathbb{R}^n$ is a metric space, the notions of distance and limit points are intuitive. How can we define limit points in the setting of an abstract topological space?

Definition 1.4. If $S$ is a topological space, then $p \in S$ is a limit point of $X \subseteq S$ if for all $U \in \mathcal{U}$ with $p \in U$,

$$(U \setminus p) \cap X \neq \emptyset$$

The closure $\overline{X}$ of $X$ is defined as $X \cup \{q \mid q$ is a limit point of $X\}$.

Definition 1.5. A collection $\mathcal{B}$ of subsets of $S$ is a basis for the topology if every finite intersection of elements of $\mathcal{B}$ is a union of elements of $\mathcal{B}$. It is not hard to show that the collection of (arbitrary) unions of elements of $\mathcal{B}$ is a topology, called the topology generated by $\mathcal{B}$.

Example 1.6. In $\mathbb{R}^n$, $\mathcal{B} = \{N_\delta(p) \mid p \in \mathbb{R}^n$ and $\delta \in \mathbb{R}_{>0}\}$ is a basis.

Our last definitions involve properties of coverings and continuity:

Definition 1.7. A collection $\mathcal{U}$ of subsets of $S$ is a cover of $S$ if

$$\bigcup_{U_i \in \mathcal{U}} U_i = S$$

$S$ is compact if every open covering admits a finite subcover, and connected if the only sets that are both open and closed are $\emptyset$ and $S$.

Example 1.8. [Heine-Borel] A set $S \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.

Definition 1.9. Let $S$ and $T$ be topological spaces. A function

$$f : S \to T$$

is continuous if $f^{-1}(U)$ is open in $S$ for every open set $U \subseteq T$. We say that $S$ and $T$ are homeomorphic if $f$ and $f^{-1}$ are 1:1 and continuous. Two maps $f_0, f_1 : S \to T$ are homotopic if there is a continuous map $F$ such that

$$S \times [0, 1] \xrightarrow{F} T$$

with $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. $F$ deforms the map $f_0$ to the map $f_1$. The spaces $S$ and $T$ are homotopic if there is a map $g : T \to S$ such that $f \circ g$ is homotopic to $1_T$ and $g \circ f$ is homotopic to $1_S$.

Exercise 1.10. Prove that if

$$f : S \to T$$

is continuous and $X$ is a compact subset of $T$, then $f^{-1}(X)$ is a compact subset of $S$. 

1.3 Discrete, Product, and Quotient topologies. We now discuss some common topologies, as well as how to build new topological spaces from old. Our first example is

**Definition 1.11.** [Discrete Topology] The topology in which every subset \(A\) of \(S\) is open is called the *discrete topology*. Since the complement \(S \setminus A\) is also a subset, this means every subset is both open and closed.

From the perspective of data science, point cloud data only comes equipped with the discrete topology; by adding a parameter we are able to bring interesting topological features into the picture in Chapter 7 with persistent homology.

**Definition 1.12.** [Product Topology]. If \(X_1\) and \(X_2\) are topological spaces, then the Cartesian product has a natural topology associated with it, the *product topology*, where open sets are unions of sets of the form \(U_1 \times U_2\) with \(U_i\) open in \(X_i\). In particular, if \(B_i\) is a basis for \(X_i\), then taking the \(U_i\) to be members of \(B_i\) shows that \(B_1 \times B_2\) is a basis for \(X_1 \times X_2\).

**Exercise 1.13.** Show that the product topology is the coarsest topology—the topology with the fewest open sets—such that the projection maps

\[
X_1 \times X_2 \xrightarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2
\]

are continuous, where \(\pi_i(x_1, x_2) = x_i\).

**Definition 1.14.** [Induced and Quotient Topology]

- If \(Y \hookrightarrow X\) then the *induced topology* on \(Y\) is the topology where open sets of \(Y\) are \(i^{-1}(U)\) for \(U\) open in \(X\).
- Let \(X\) be a topological space and \(\sim\) an equivalence relation. The quotient space \(Y = X/\sim\) is the set of equivalence classes, endowed with the *quotient topology*: \(U \subseteq Y\) is open iff
  \[
  V = \pi^{-1}(U) \text{ is open in } X, \text{ where } X \xrightarrow{\pi} X/\sim.
  \]

**Example 1.15.** For \(Y \subseteq X\), define an equivalence relation via

\[
a \sim b \iff a, b \in Y
\]

This collapses \(Y\) to a point, so for example if \(X = [0, 1]\) with the induced topology from \(\mathbb{R}^1\) and \(Y\) consists only of the two endpoints \(\{0, 1\}\) then \(X/Y \simeq S^1\). If instead \(X\) is the unit square \([0, 1] \times [0, 1]\), if \(Y = \partial(X)\) then \(X/Y\) is the two sphere \(S^2\), whereas if \(Y = (x, 1) \sim (x, 0)\) and \((0, y) \sim (1, y)\) then \(X/Y \simeq T^2\).
2. Vector Bundles

While vector bundles are fundamental objects in topology, the reason to include them in these notes is that they provide the right intuition to motivate the definition of a sheaf. In this section we work with real, topological vector bundles, where maps between objects are continuous, written as $C^0$; for algebraic vector bundles see [34]. The formal definition appears below; the intuition is that a rank $n$ vector bundle $\mathcal{V}$ is locally a product space of the form

$$ U_i \times \mathbb{R}^n, $$

along with projection maps from $\mathcal{V} \to U_i$. We will also touch briefly on geometry to discuss the tangent and cotangent bundles of a manifold, where the maps between objects will be differentiable, written as $C^\infty$.

**Example 2.1.** A product space of the form $X \simeq Y \times \mathbb{R}^n$ is a trivial vector bundle; for example an infinite cylinder $C$ is

$$ C \simeq S^1 \times \mathbb{R}. $$

A small connected open set $U \subseteq S^1$ satisfies $U \simeq \mathbb{R}$, and $C$ can be covered by sets of this form. On the other hand, the Möbius band $M$ is also a rank one vector bundle on $S^1$, and locally has the same form as $C$. As Exercise 2.5 shows, the spaces $M$ and $C$ are different.

**Definition 2.2.** A topological space $\mathcal{V}$ is a rank $n$ vector bundle (typically called the total space) on a topological space $X$ (typically called the base space) if there is a continuous map

$$ \mathcal{V} \xrightarrow{\pi} X $$

such that

- There exists an open cover $U_i$ of $X$, and homeomorphisms $\phi_i$

  $$ \pi^{-1}(U_i) \xrightarrow{\phi_i} U_i \times \mathbb{R}^n $$

  such that $\phi_i$ followed by projection onto $U_i$ is $\pi$, and

- The transition maps $\phi_{ij} = \phi_i \circ \phi_j^{-1}$:

  $$ \phi_j \pi^{-1}(U_j) \supseteq (U_i \cap U_j) \times \mathbb{R}^n \xrightarrow{\phi_{ij}} (U_i \cap U_j) \times \mathbb{R}^n \subseteq \phi_i \pi^{-1}(U_i) $$

  satisfy $\phi_{ij} \in GL_n(C^0(U_i \cap U_j))$.

A cover $\{U_i\}$ so that the first bullet above holds is a trivialization, because locally a vector bundle on $X$ looks like a product of $U_i$ with $\mathbb{R}^n$. The map $\phi_j$ gives us a chart: a way to think of a point in $\pi^{-1}(U_j)$ as a point $(t, v) \in U_j \times \mathbb{R}^n$. The copy of $\mathbb{R}^n \simeq \phi_i \pi^{-1}(p)$ over a point $p$ is called the fiber over $p$. If we change from a chart over $U_j$ to a chart over $U_i$, then

$$ (t, v_j) \mapsto (t, [\phi_{ij}(t)] \cdot v_j) $$
We’ve seen this construction in the first chapter, in the section on change of basis. Suppose \( B_1 \) and \( B_2 \) are different bases for a vector space \( V \), with \([v]_{B_i}\) denoting the representation of \( v \in V \) with respect to basis \( B_i \). Then if \( \Delta_{ij} \) denotes the transition matrix from \( B_j \) to \( B_i \), we have
\[
[v]_{B_2} = \Delta_{21}[v]_{B_1}
\]
The \( \phi_{ij} \) are transition functions, and are a family of transition matrices, which vary as \( p \in U_i \) varies. A vector space does not have a canonical basis, and this is therefore also true for the fibers of a vector bundle. It follows from the definition that \( \phi_{ik} = \phi_{ij} \circ \phi_{jk} \) on \( U_i \cap U_j \cap U_k \) and that \( \phi_{ii} \) is the identity.

**Definition 2.3.** A section of a vector bundle \( \mathcal{V} \) over an open set \( U \subseteq X \) is a morphism
\[
U \xrightarrow{s} \mathcal{V}
\]
such that \( \pi \circ s = 1 \).
Let \( \mathcal{S}(U) \) denote the set of all sections of \( \mathcal{V} \) over \( U \). A map from \( U \to \mathbb{R}^1 \) is given by a continuous function \( f \in C^0(U) \), so a map from \( U \) to \( \mathbb{R}^n \) is given by an \( n \)-tuple of continuous functions, and is therefore an element of \( C^0(U)^n \). This means that locally \( \mathcal{S}(U) \) has the structure of a free module over the ring \( C^0(U)^n \) of continuous functions on \( U \).

**Exercise 2.4.** Use properties of the transition functions and continuity to show that if \( s_j = (f_1, \ldots, f_n) \in \mathcal{S}(U_j) \) and \( s_i = (g_1, \ldots, g_n) \in \mathcal{S}(U_i) \) are sections such that
\[
\phi_{ij}(s_j|_{U_i \cap U_j}) = s_i|_{U_i \cap U_j},
\]
then there exists \( s \in \mathcal{S}(U_i \cup U_j) \) which restricts to the given sections. When \( \mathcal{S}(U_i) \) is a free module over some ring of functions on all sufficiently small open sets \( U_i \), then \( \mathcal{S} \) is called a locally free sheaf.

**Exercise 2.5.** Let \( S \subseteq \mathbb{R}^2 \) be a smooth surface, \( p \in S \) and \( p(t), t \in [0, 1] \) a smooth loop on \( S \) with \( p(0) = p = p(1) \). If \( n(p(0)) \) is a normal vector to \( S \) at \( p \) and \( p(t) \) goes thru charts \( U_i, i \in \{1, \ldots, n\} \) and \( \phi_{i+1,i} \) are the transition functions from \( U_i \) to \( U_{i+1} \), then \( S \) is orientable if \( \phi_1 \cdots \phi_2 n(p(0)) = n(p(1)) \). Show a cylinder is orientable but the Möbius strip is not.

### 3. Sheaf Theory

We introduced vector bundles in the previous section because they give a good intuition for sheaves. For vector bundles, on any open set the fibers over the base are the same; this is not the case for an arbitrary sheaf. But the key concept is the same: a sheaf is defined by local data, together with information that stipulates how to transition from one coordinate system (known as a chart) to another.

**Definition 3.1.** Let \( X \) be a topological space. A presheaf \( \mathcal{F} \) on \( X \) is a rule associating to each open set \( U \subseteq X \) an algebraic object \( \mathcal{F}(U) \):
\[
U \to \mathcal{F}(U),
\]

together with the conditions

(a) If \( U \subseteq V \) there is a homomorphism
\[
\mathcal{F}(V) \xrightarrow{\rho_{VU}} \mathcal{F}(U).
\]

(b) \( \mathcal{F}(\emptyset) = 0 \).

(c) \( \rho_{UU} = \text{id}_{\mathcal{F}(U)} \).

(d) If \( U \subseteq V \subseteq W \) then
\[
\rho_{VV} \circ \rho_{WV} = \rho_{WU}.
\]

The map \( \rho_{VU} \) is called restriction. For \( s \in \mathcal{F}(V) \) we write \( \rho_{VU}(s) = s|_U \).
A sheaf is a presheaf which satisfies two additional properties.

**Definition 3.2.** A presheaf $\mathcal{F}$ on $X$ is a sheaf if for any open set $U \subseteq X$ and open cover $\{V_i\}$ of $U$,

(a) If $s \in \mathcal{F}(U)$ satisfies $s|_{V_i} = 0$ for all $i$, then $s = 0$ in $\mathcal{F}(U)$.

(b) For $v_i \in \mathcal{F}(V_i)$ and $v_j \in \mathcal{F}(V_j)$ such that $v_i|_{(V_i \cap V_j)} = v_j|_{(V_i \cap V_j)}$, there exists $t \in \mathcal{F}(V_i \cup V_j)$ such that $t|_{V_i} = v_i$ and $t|_{V_j} = v_j$.

The second condition is referred to a gluing: when elements on sets $U_1$ and $U_2$ agree on $U_1 \cap U_2$, they can be glued to give an element on $U_1 \cup U_2$.

**Example 3.3.** A main source of examples comes from functions on $X$. When $X$ has more structure, for example, when $X$ is a real or complex manifold, then the choice of functions will typically reflect this structure.

- $X$: topological space
- $\mathcal{F}(U)$: $C^0$ functions on $U$.
- $X$ a smooth real manifold: $C^\infty$ functions on $U$.
- $X$ a complex manifold: holomorphic functions on $U$.

The $\mathcal{F}(U)$ above are all rings, which means in any of those situations it is possible to construct sheaves $\mathcal{M}$, where $\mathcal{M}(U)$ is a $\mathcal{F}(U)$-module. Notice how this fits with Exercise 2.4 on the sections of a topological vector bundle.

**Example 3.4.** For a ring of the form $R = \mathbb{K}[x_1, \ldots, x_n]/I$, the module of $\mathbb{K}$-derivations $\text{Der}_\mathbb{K}(R)$ is the set of $\mathbb{K}$-linear maps $D : R \to R$ such that $D(r_1 r_2) = r_1 D(r_2) + r_2 D(r_1)$. Suppose $\mathbb{K}$ is algebraically closed, so that a point $p$ corresponds to a maximal ideal $m_p$. Then $\text{Der}_\mathbb{K}(R)_{m_p}$ corresponds to the tangent space $T_p(X)$. The tangent sheaf is defined in Example 4.3.

One of the earliest uses of sheaf theory was in complex analysis: the Mittag-Leffler problem involves constructing a meromorphic function with residue 1 at all integers. Clearly in some small neighborhood of $k \in \mathbb{Z}$

$$\frac{1}{z - k}$$

works, so the question is how to glue things together. Consider a one dimensional compact complex manifold $C$, which corresponds topologically to a smooth compact real surface of genus $g$. On a small open subset corresponding to a complex disk, there are plenty of holomorphic functions. But trying to glue them together to give a global nonconstant holomorphic function is doomed to failure.

**Exercise 3.5.** For those who remember complex analysis: the maximum modulus principal implies that if $f(z)$ is a holomorphic function on a bounded domain $D$, then $|f(z)|$ attains it’s maximum on $\partial(D)$. Show this implies that there are no nonconstant holomorphic functions on $C$. ◊
3. Sheaf Theory

3.1. Posets, Direct Limit, and Stalks. Sheaves carry local data. The previous discussion involved the complex numbers, where the topological structure arises automatically from the fact that \( \mathbb{C} \) is a metric space. For \( p \in \mathbb{C} \), functions \( f \) and \( g \) have the same germ at \( p \) when their Taylor series expansions at \( p \) agree, which allows a gluing construction. In the setting of persistent and multiparameter persistent homology, the sheaves we encounter will involve the polynomial ring \( R = \mathbb{K}[x_1, \ldots, x_n] \). The fundamental topology used in this context is the Zariski topology.

Definition 3.6. The Zariski Topology on \( \mathbb{K}^n \) has as closed sets \( V(I) \), ranging over all ideals \( I \subseteq R \). Clearly \( V(1) = \emptyset \) and \( V(0) = \mathbb{K}^n \). The corresponding topological space is known as affine space, written as \( \mathbb{A}^n_{\mathbb{K}} \) or Spec(\( R \)). As \( V(I) \subseteq \mathbb{A}^n_{\mathbb{K}} \), it inherits the induced topology appearing in Definition 1.14.

Exercise 3.7. Use Exercise 4.7 of Chapter 2 to show that Definition 3.6 does define a topology. A topological space is Noetherian if there are no infinite descending chains of closed sets. Prove the Zariski topology is Noetherian.

In the classical topology on \( \mathbb{R}^n \), the notion of smaller (and smaller) \( \varepsilon \)-neighborhoods of a point is clear and intuitive. In the Zariski topology, all open sets are complements of varieties, so are dense in the classical topology—the open sets are very big. What is the correct analog of a small open set in this context? The solution is a formal algebraic construction, known as the direct limit.

Definition 3.8. A directed set \( S \) is a partially ordered set with the property that if \( i, j \in S \) then there exists \( k \in S \) with \( i \leq k, j \leq k \). Let \( R \) be a ring and \( \{M_i\} \) a collection of \( R \)-modules, indexed by a directed set \( S \), such that for each pair \( i \leq j \) there exists a homomorphism \( \mu_{ji} : M_i \to M_j \). If \( \mu_{ii} = 1_{M_i} \) for all \( i \) and \( \mu_{kj} \circ \mu_{ji} = \mu_{ki} \) for all \( i \leq j \leq k \), then the modules \( M_i \) are said to form a directed system. Given a directed system, the direct limit is an \( R \)-module constructed as follows: let \( N \) be the submodule of \( \oplus M_i \) generated by the relations \( m_i - \mu_{ji}(m_i) \), for \( m_i \in M_i \) and \( i \leq j \). Then the direct limit is

\[
\lim_{\longrightarrow} M_i = (\bigoplus_{i \in S} M_i)/N.
\]

To phrase it a bit differently: elements \( m_i \in M_i \) and \( m_j \in M_j \) are identified in the direct limit if the images of \( m_i \) and \( m_j \) eventually agree.

Definition 3.9. The stalk of a sheaf at a point is the direct limit over all open sets \( U \) containing \( p \):

\[
\mathcal{F}_p = \lim_{\longrightarrow} \mathcal{F}(U).
\]

The stalk is obtained by taking all open neighborhoods \( U_i \) of \( p \), and then using the restriction maps to identify elements which agree on a small enough neighborhood. This mirrors the construction of the germ of a function discussed above: if
$\mathcal{H}(U)$ is the sheaf of holomorphic functions on $U \subseteq \mathbb{C}$ in the classical topology, then given

$$f_1 \in \mathcal{H}(U_1) \text{ and } f_2 \in \mathcal{H}(U_2)$$

such that $p \in U_1$, then $f_1 = f_2$ in the stalk $\mathcal{H}_p$ if they have the same Taylor series expansion at $p$. The stalk at $p$ consists of convergent power series at $p$. In particular, the stalk encodes very local information. Because stalks represent very local data, they serve as the building blocks for algebraic constructions with sheaves.

### 3.2. Morphisms of Sheaves and Exactness.

When a sheaf is given locally by an algebraic object, standard algebraic constructions will have sheaf-theoretic analogs. However, there are usually some subtle points to deal with, as illustrated below.

**Definition 3.10.** A morphism of sheaves

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G}$$

on $X$ is defined by giving, for all open $U$, maps

$$\mathcal{F}(U) \xrightarrow{\phi(U)} \mathcal{G}(U),$$

which commute with the restriction map.

- $\phi$ is injective if for any $p \in X$ there exists an open set $U$ with $\phi(U)$ injective (which implies $\phi(V)$ is injective for all $V \subseteq U$).
- $\phi$ is surjective if for any $p \in X$ and $g \in \mathcal{G}(U)$ there exists $V \subseteq U$ with $p \in V$ and $f \in \mathcal{F}(V)$ such that $\phi(V)(f) = p_{UV}(g)$.

For a morphism of sheaves as above, the assignment $U \mapsto \ker(\phi)(U)$ defines a sheaf. However, the image and cokernel of $\phi$ are only presheaves, and it is necessary to do a bit of preparatory work, by building a *sheaf associated to a presheaf*, to obtain the sheaf-theoretic version of image and cokernel. A sequence of sheaves and sheaf homomorphisms

$$\cdots \rightarrow \mathcal{F}_{i+1} \xrightarrow{d_{i+1}} \mathcal{F}_i \xrightarrow{d_i} \mathcal{F}_{i-1} \rightarrow \cdots$$

is exact when the kernel sheaf of $d_i$ is equal to the image sheaf of $d_{i+1}$. For proofs of the theorem below, as well as for the construction of the sheaf associated to a presheaf mentioned above, see Chapter II of [62].

**Theorem 3.11.** A sequence of sheaves is exact iff it is exact at the level of stalks.

At the start of this chapter, we noted that sheaves are bookkeeping devices which encode when objects glue together, as well as when there are obstructions. The *global sections* of $\mathcal{F}$ are globally defined elements of $\mathcal{F}$, and are written as $\mathcal{F}(X)$ or $\Gamma(\mathcal{F})$ or $\check{H}^0(\mathcal{F})$. In Chapter 6 we define Čech cohomology, which is a homology theory specifically designed to measure the conditions (and obstructions) involved in gluing local sections together.
4. Sheaves on Algebraic Varieties

We saw in §2 that vector bundles are topological spaces that are easy to visualize, at least locally. Is it possible to visualize a sheaf? The answer is yes, and the intuition comes from vector bundles. A rank one vector bundle on a curve might look like

The gluing axioms for sheaves satisfy the same properties required to define transition functions for vector bundles. For vector bundles the base space \( B \) may be complicated, but the fibers are all simply vector spaces. For sheaves on varieties, the base space will be an affine variety as in Definition 3.6. Our main focus will be on sheaves on \( \mathbb{A}^n_K \) itself, which give the right intuition for the general case.

**Definition 4.1.** As a set, an affine variety is simply \( \mathbb{V}(I) \). Since \( \mathbb{V}(I) \subseteq \mathbb{A}^n_K \), it has the induced Zariski topology from \( \mathbb{A}^n \). On \( \mathbb{A}^n \) regular functions are polynomials in \( R = K[x_1, \ldots, x_n] \), and since \( f \in I \) is the zero function on \( \mathbb{V}(I) \), regular functions on \( \mathbb{V}(I) \) are polynomials defined up to \( I \), that is, elements of \( R/I \).

The fundamental sheaf in algebraic geometry is the sheaf of regular functions.

**Definition 4.2.** On an affine variety \( X = \mathbb{V}(I) \subseteq \mathbb{A}^n \) and Zariski open set \( U \), the sheaf of regular functions is defined as

\[
\mathcal{O}_X(U) = \left\{ \frac{f}{g} \mid f, g \in R/I \right\},
\]

with \( g(p) \neq 0 \) for \( p \in U \); since \( U \) is open, \( U = \mathbb{V}(J)^c \) for some ideal \( J \). By the Noetherian condition, the sets \( U_f = \mathbb{V}(f)^c \) for all \( f \in R \) are a basis for the Zariski topology, and any \( \mathcal{M} \) which is a sheaf of \( \mathcal{O}_X \)-modules can be defined by starting with an ordinary \( R \)-module \( M \), and defining \( \mathcal{M} \) on the basis via

\[
\mathcal{M}(U_f) = M_f, \quad \text{with the stalks corresponding to } \mathcal{M}_p = M_p,
\]

where \( M_f \) and \( M_p \) are obtained from the localization operation of Chapter 2.

**Definition 4.3.** The tangent sheaf is constructed from the modules in Example 3.4:

\[
\mathcal{T}_X(U_i) = \text{Der}_K \mathcal{O}_X(U_i)
\]

A point \( p \in X \) is smooth if the stalk \( \mathcal{T}_{X,p} \) is a free \( \mathcal{O}_{X,p} \)-module; \( X \) is smooth if every point is a smooth point. This mirrors the construction of the tangent bundle of an \( n \)-dimensional real manifold \( M \). In this situation, \( K = \mathbb{R} \) and there is a cover of \( M \) by open sets \( U_i \) diffeomorphic to \( \mathbb{R}^n \). The transition maps \( \phi_{ij} \) appearing in Definition 2.2 are \( C^\infty \) maps between charts, and the fibers are \( \mathbb{R}^n \).
Definition 4.4. The support of an $R$-module $M$ consists of the prime ideals $P$ such that $M_P \neq 0$.

When $M$ is a finitely generated $R$-module, a prime ideal $P$ is in the support of $M$ iff $\text{ann}(M) \subseteq P$. If $M$ is generated by $\{m_1, \ldots, m_k\}$, then there is a surjection $R^k \to M \to 0$.

Localizing at a prime $P$ makes elements of $R$ outside $P$ invertible, therefore if $\text{ann}(M) \nsubseteq P$, then some element $r \in \text{ann}(M)$ is a unit in $R_P$. But a unit can annihilate every element of $M_P$ only if $M_P = 0$.

4.1. Associated Primes and Primary Decomposition. We defined the variety of an ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ in Chapter 2:

$$V(I) = \{p \in \mathbb{K}^n \mid f(p) = 0 \text{ for all } f \in I\}$$

Example 4.5. For $I = \langle x^2 - x, xy - x \rangle \subseteq \mathbb{R}[x, y]$, $V(I) \subseteq \mathbb{R}^2$ is pictured below:

So $V(I)$ consists of the line $V(x)$ and the point $V(x - 1, y - 1)$. Notice that

$$I = \langle x \rangle \cap \langle x - 1, y - 1 \rangle$$

By Theorem 2.10 in Chapter 2, $I$ is prime iff $R/I$ is a domain, and since

$$\mathbb{R}[x, y]/\langle x \rangle \simeq \mathbb{R}[y] \text{ and } \mathbb{R}[x, y]/\langle x - 1, y - 1 \rangle \simeq \mathbb{R}$$

are both domains, we see that $I$ is the intersection of two prime ideals.

It is clearly too much to hope that any ideal is an intersection of prime ideals:

$$\langle x^2 \rangle \subseteq \mathbb{R}[x]$$

cannot be written as an intersection of prime ideals. But it turns out that any ideal in a Noetherian ring has a decomposition as a finite intersection of irreducible ideals, which appeared in Definition 2.8 in Chapter 2. Recall an ideal $I$ is irreducible if it is not the intersection of two proper ideals, each strictly containing $I$.

**Theorem 4.6.** In a Noetherian ring $R$, any ideal $I$ can be written as

$$I = \bigcap_{i=1}^{n} I_i,$$

with $I_i$ irreducible.
Proof. Let $\Sigma$ be the set of all ideals which cannot be written as a finite intersection of irreducible ideals. Since $R$ is Noetherian, $\Sigma$ has a maximal element $I'$. Since $I'$ is reducible,

$$I' = J_1 \cap J_2$$

with $I' \subseteq J_1$ and $I' \subseteq J_2$. Since $I'$ is maximal, neither $J_1$ nor $J_2$ are in $\Sigma$. Hence both $J_1$ and $J_2$ can be written as finite intersections of irreducibles, thus $I'$ is a finite intersection of irreducibles, a contradiction. \qed

Primary ideals also appeared in Definition 2.8 in Chapter 2; $Q$ is primary if

$$xy \in Q \Rightarrow x \in Q \text{ or } y^n \in Q \text{ for some } n.$$ 

In a Noetherian ring an irreducible ideal is primary ([88] Lemma 1.3.3), so we have

**Theorem 4.7.** In a Noetherian ring $R$, any ideal $I$ can be written as

$$I = \bigcap_{i=1}^{n} Q_i,$$

with $Q_i$ primary.

It follows immediately from the definition that the radical of a primary ideal is prime, since $x$ or $y^n \in Q \Rightarrow x$ or $y \in \sqrt{Q}$.

**Definition 4.8.** For a primary decomposition

$$I = \bigcap_{i=1}^{n} Q_i$$

the $P_i = \sqrt{Q_i}$ appearing in this decomposition are the Associated Primes of $I$.

**Exercise 4.9.** Show primary decomposition is not unique:

$$\langle xy, x^2 \rangle = \langle x^2, y \rangle \cap \langle x \rangle = \langle x^2, xy, y^2 \rangle \cap \langle x \rangle$$

In particular, show $\langle x^2, y \rangle$ and $\langle x^2, xy, y^2 \rangle$ are primary. What are the associated primes? \hfill \diamond

There is a version of associated primes for an $R$-module $M$:

**Definition 4.10.** A prime ideal $P$ is associated to an $R$-module $M$ if

$$P = \text{ann}(m) \text{ for some } m \in M,$$

where an element $r \in R$ is in the ideal $\text{ann}(m)$ if $r \cdot m = 0.$
Notice that any $m \in M$ generates a principal submodule $R \cdot m \subseteq M$, so we have a diagram as below, where the vertical arrow is an isomorphism.

$$
\begin{array}{c}
0 \\
\downarrow \\
R/\text{ann}(M) \\
\end{array} \\
\xrightarrow{\text{iso}} \\
\begin{array}{c}
R \cdot m \\
M
\end{array}
$$

**Exercise 4.11.** Show that when $M = R/I$ and $I = \cap Q_i$, then for any index $i$

$$I \subseteq Q_i \subseteq P_i$$

and it is possible to find $m \in M$ so that $\text{ann}(m) = P_i$. So in the setting of a cyclic module $M = R/I$, Definition 4.10 and Definition 4.8 agree. ◊

**Example 4.12.** We return to the example from the beginning of the section, where $R = \mathbb{R}[x, y]$ and

$$I = \langle x^2 - x, xy - y \rangle.$$  

Let $X = \mathbb{A}^2$, so that $R/I$ defines a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules. From the definition of the support of a module, to visualize the stalks $\mathcal{M}_P$, we need to localize at the associated primes.

- If $P = \langle x - 1, y - 1 \rangle$, then $x \notin P$ so $x$ is a unit, and

  $$(R/I)_P \simeq R_P/I_P \simeq R_P/(x - 1, y - 1)_P \simeq (\mathbb{R})_P \simeq \mathbb{R}.$$  

  so the stalk $\mathcal{M}_P$ at the point $(1, 1)$ is just a point.

- If $P = \langle x \rangle$, then $\{x - 1, y - 1\}$ are both units, and

  $$(R/I)_P \simeq R_P/I_P \simeq R_P/\langle x \rangle_P \simeq \mathbb{R}[y]_P.$$  

  so the stalk $\mathcal{M}_P$ along the line $V(x)$ is a line.

- If $P$ is a prime $\notin \{\langle x \rangle, \langle x - 1, y - 1 \rangle\}$, let $P$ be an ideal of the form

  $$\langle x - a, y - b \rangle \text{ with } (a, b) \in \mathbb{R}^2 \setminus \{V(x), (1, 1)\}.$$  

  Then $y - 1$ is a unit in $R_P$, and $x$ is a unit in $R_P$, so $xy - x \notin P$. Hence $xy - x$ is a unit, which also annihilates $\mathcal{M}$ so $\mathcal{M}_P = 0$.

This shows that $\mathcal{M}$ is zero everywhere except along the line $V(x)$ and at the point $V(x - 1, y - 1)$:
Chapter 5

Homology I: From Simplicial Theory to Sensor Networks

This chapter is the fulcrum of the book, the beginning of a beautiful friendship between algebra and topology. The idea is the following: small open neighborhoods of a point on the circle $S^1$ and a point on the line $\mathbb{R}^1$ are homeomorphic—an ant living in either neighborhood would be unable to distinguish between them. But from a global perspective we can easily spot the difference between $\mathbb{R}^1$ and $S^1$. Homology provides a way to quantify and compute the difference.

The first step is to model a topological space combinatorially. We do this with simplicial complexes, which are a lego set for topologists. The second step is to translate the combinatorial data of a simplicial complex into an algebraic complex, as in Definition 2.5 of Chapter 2. Simplicial homology is the homology of the resulting complex; we also discuss singular homology, which is based on formal combinations of maps from a simplex into a topological space.

In §3, we introduce CW complexes and cellular homology; the building blocks are cells and allow more flexibility than simplicial complexes in construction of spaces. In §4 we tackle duality, focusing on the theories of Poincaré and Alexander. Poincaré duality is like the Tai Chi symbol—the Yin and Yang are symmetric mirrors. Alexander duality is like a black and white image: knowing one color tells us the other; TDA applications appear in [1] and [57]. The roadmap for this chapter is:

- Simplicial Complexes.
- Simplicial and Singular Homology.
• Fundamentals: Snake Lemma, Long Exact Sequence in Homology.
• Applications: Mayer-Vietoris, Relative Homology and Evasion problems

1. Simplicial Complexes

Simplices are natural candidates for building blocks in topology:

\[
\begin{array}{c|c|c|c}
0 \text{-simplex} & 1 \text{-simplex} & 2 \text{-simplex} & 3 \text{-simplex} \\
\text{vertex} & \text{edge} & \text{triangle} & \text{tetrahedron}
\end{array}
\]

Simplices may be thought of as combinatorial or geometric objects:

**Definition 1.1.** An \( n \)-simplex \( \sigma_n \) is an \( n+1 \) set and all subsets. A **geometric realization** of an \( n \)-simplex is the convex hull (set of all convex combinations) \( C \) of \( n+1 \) points in \( \mathbb{R}^d \), such that \( C \) is \( n \)-dimensional (so \( d \geq n \)). The **standard** \( n \)-simplex is the convex hull of the coordinate points in \( \mathbb{R}^{n+1} \).

A simplicial complex \( \Delta \) is constructed by attaching a group of simplices to each other, with the caveat that the attaching is done by identifying shared sub-simplices; sub-simplices are often called **faces**.

**Definition 1.2.** An abstract simplicial complex \( \Delta \) on vertex set \( V \) is a collection of subsets of \( V \) such that

- \( v \in \Delta \) if \( v \in V \).
- \( \tau \subseteq \sigma \in \Delta \Rightarrow \tau \in \Delta \).

**Example 1.3.** The second condition above says that to specify a simplicial complex, it suffices to describe the maximal faces. For example, we visualize the simplicial complex on \( V = \{0, 1, 2, 3\} \) with maximal faces \( \{0, 1, 2\} \) and \( \{1, 3\} \) as a triangle with vertices labelled \( \{0, 1, 2\} \) and an edge with vertices labelled \( \{1, 3\} \). For a non-example, consider the same triangle, but where the vertex labelled 1 from edge \( \{1, 3\} \) is glued to the middle of the edge \( \{1, 2\} \).
Definition 1.4. The dimension of $\Delta$ is the dimension of the largest face of $\Delta$. An orientation on $\Delta$ is a choice of ordering of the vertices of $\sigma$ for each $\sigma \in \Delta$. Oriented one and two simplices are depicted below.

We will use square brackets to denote oriented simplices.

2. Simplicial and Singular Homology

We recall the definition of an algebraic complex from Chapter 2: a complex $\mathcal{C}$ is a sequence of modules and homomorphisms

$$\mathcal{C}_\bullet : \cdots \to M_{j+1} \xrightarrow{d_{j+1}} M_j \xrightarrow{d_j} M_{j-1} \cdots$$

such that $\text{im}(d_{j+1}) \subseteq \text{ker}(d_j)$; the $i^{th}$ homology is $H_i(\mathcal{C}) = \ker(d_j)/\text{im}(d_{j+1})$.

Definition 2.1. Let $\Delta$ be an oriented simplicial complex, and let $R$ be a ring. Define $C_i(\Delta, R)$ as the free $R$-module with basis the oriented $i$-simplices, modulo the relations

$$\{v_0, \ldots, v_n\} \sim (-1)^{\text{sgn}(\sigma)}[v_{\sigma(0)}, \ldots, v_{\sigma(n)}],$$

where $\sigma \in S_{n+1}$ is a permutation. Any permutation $\sigma$ can be written as a product of transpositions (two cycles); the sign of $\sigma$ is one if the number of transpositions is odd, and zero if the number of transpositions is even.

Example 2.2. For an edge, this says

$$[v_0, v_1] = -[v_1, v_0]$$

This is familiar from vector calculus: when it comes to integration

$$\int_{v_0}^{v_1} f(x)dx = -\int_{v_1}^{v_0} f(x)dx$$

For a triangle, the orientation $[v_2, v_1, v_0]$ requires 3 transpositions to change to $[v_0, v_1, v_2]$, hence

$$[v_2, v_1, v_0] = -[v_0, v_1, v_2]$$

Definition 2.3. Let $\partial$ be the map from $C_i(\Delta, R)$ to $C_{i-1}(\Delta, R)$ defined by

$$\partial([v_{j_0}, \ldots, v_{j_i}]) = \sum_{k=0}^{i} (-1)^k [v_{j_0}, \ldots, \hat{v}_{j_k}, \ldots, v_{j_i}]$$

Applying $\partial^2$ to a basis element shows $\partial^2 = 0$, so the $C_i$ form a chain complex $\mathcal{C}$. Simplicial homology $H_i(\Delta, R)$ is the homology of $\mathcal{C}$. If we augment $\mathcal{C}$ by adding in the empty face so $C_{-1}(\Delta, R) = R$, the result is reduced homology $\tilde{H}_i(\Delta, R)$.
Example 2.4. If $R$ is a field, the $C_i$ are a complex of vector spaces.

$$\mathcal{V} : 0 \rightarrow V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_0 \rightarrow 0.$$ 

An induction shows that

$$\sum_{i=0}^{n} (-1)^i \dim V_i = \sum_{i=0}^{n} (-1)^i \dim H_i(\mathcal{V}).$$

The alternating sum $\chi(\mathcal{V})$ is known as the Euler characteristic.

Example 2.5. We return to our pair of ants, one living on $S^1$ and one on $R^1$. Let $K$ be a field, and model $S^1$ and $R^1$ with oriented simplicial complexes $\Delta_1$ and $\Delta_2$ having maximal faces

$$\Delta_1 = \{[0,1], [1,2], [2,0]\} \text{ and } \Delta_2 = \{[0,1], [1,2], [2,3]\}$$

Let $C_1(\Delta_i, K)$ have the ordered bases above, and the ordered basis of $C_0(\Delta_1, K)$ be $\{[0], [1], [2]\}$. The only nonzero differential is $\partial_1$:

$$C_\bullet(\Delta_1, K) = 0 \rightarrow K^3 \xrightarrow{\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}} K^3 \rightarrow 0$$

and we compute that $H_1(\Delta_1) = \ker(\partial_1) = [1,1,1]^T$. On the other hand, for $\Delta_2$, using $\{[0], [1], [2], [3]\}$ as basis for $C_0(\Delta_2)$, the complex is

$$C_\bullet(\Delta_2, K) = 0 \rightarrow K^3 \xrightarrow{\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}} K^3 \rightarrow 0$$

and we compute that $H_1(\Delta_2) = 0$. So for this example, the first homology module $H_1$ distinguishes between $\Delta_1$ and $\Delta_2$.

Exercise 2.6. The example above illustrates the key point about homology: $H_k$ measures $k$-dimensional holes. Draw a picture and compute homology when $\Delta$ has oriented simplices $\{[0,1], [1,2], [2,0], [2,3], [0,3]\}$.

Exercise 2.7. Prove $\text{rank}(H_0(\Delta, R)) = \sharp$ of connected components of $\Delta$.

We can also define elements of $C_i(\Delta, R)$ as functions $c$ from the $i$-simplices to $R$ which are zero except on finitely many simplices, such that if $\tau$ and $\tau'$ are the same as unoriented simplices and $\tau = \sigma(\tau')$ then $c(\tau) = \text{sgn}(\sigma)c(\tau')$. This is useful for dealing with infinite simplicial complexes.

Is simplicial homology a topological invariant? The answer is yes, but the proof is tedious. There is, however, a close cousin to simplicial homology, known as singular homology, where topological invariance is automatic. The downside is that singular homology is not as computationally tractable as simplicial homology.
Example 2.8. We compute the simplicial homology of the sphere $S^2$, using two different triangulations. Let $\Delta_1$ be the simplicial complex defined by the boundary of a tetrahedron, having maximal faces $\{[1, 2, 3], [1, 2, 4], [1, 3, 4], [2, 3, 4]\}$, and let $\Delta_2$ be the simplicial complex consisting of two hollow tetrahedra, glued along a common triangular face, with the glued face subsequently removed; $\Delta_2$ has maximal faces $\{[1, 2, 4], [1, 3, 4], [2, 3, 4], [1, 2, 5], [1, 3, 5], [2, 3, 5]\}$.

With oriented, ordered basis for $C_2(\Delta_1, \mathbb{R})$ as above, oriented basis for $C_1(\Delta_1, \mathbb{R}) = \{[1, 2], [1, 3], [1, 4], [2, 3], [2, 4], [3, 4]\}$, and basis for $C_0(\Delta_1, \mathbb{R}) = \{[1], [2], [3], [4]\}$, the complex $C_\bullet(\Delta_1, \mathbb{R})$ is given by

$$0 \rightarrow \mathbb{R}^4 \xrightarrow{\partial_2} \mathbb{R}^6 \xrightarrow{\partial_1} \mathbb{R}^4 \rightarrow 0$$

with

$$\partial_2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

and $\ker(\partial_2) = [1, 1, 1, 1]^T$.

And

$$\partial_1 = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

We compute $\ker(\partial_1) = \text{im}(\partial_2)$, and $\text{im}(\partial_1) \simeq \mathbb{R}^3 \subseteq \mathbb{R}^4 = \ker(\partial_0)$, hence

$$H_2(\Delta_1, \mathbb{R}) \simeq \mathbb{R}^3$$
$$H_1(\Delta_1, \mathbb{R}) \simeq \mathbb{R}^3$$
$$H_0(\Delta_1, \mathbb{R}) \simeq \mathbb{R}$$

Exercise 2.9. Compute $H_i(\Delta_2, \mathbb{R})$. You should obtain the same values for homology as above: $H_1(\Delta_2, \mathbb{R})$ vanishes, and $H_2(\Delta_2, \mathbb{R}) \simeq \mathbb{R} \simeq H_0(\Delta_2, \mathbb{R})$. We’ll see in §4 that this symmetry is a consequence of Poincaré duality.

The simplest possible model for $S^2$ would consist of two triangles, glued together along the 3 common edges. The definition of CW complexes in §3 will allow this, and in fact permit an even simpler model for $S^2$. 

$\diamondsuit$
2.1. Singular homology. In singular homology, the objects of study are maps to a topological space $X$. A singular $n$-chain is a continuous map from the standard $n$-simplex $\Delta_n$ to $X$. The set of all singular $n$-chains on $X$ is denoted $S_n(X)$; it is a free $\mathbb{Z}$-module. If $\sigma \in S_n(X)$, then restricting $\sigma$ to an $n-1$ face of $\Delta_n$ yields (after an appropriate change of coordinates) an element of $S_{n-1}(X)$, and we can build a chain complex in a fashion exactly as we did with simplicial homology:

Definition 2.10. Let $r_i$ be the map that changes coordinates from the basis vectors \{e_1, \ldots, \hat{e}_i, \ldots e_{n+1}\} to the basis vectors \{e_1, \ldots, e_n\}. For $f \in S_n(X)$, let $f_i$ be the restriction of $f$ to the $i^{th}$ face of $\Delta_n$, and define a complex $\mathcal{S} : \cdots \rightarrow S_n(X) \xrightarrow{\partial} S_{n-1}(X) \rightarrow \cdots$ via

$$\partial(f) = \sum_{i=1}^{n+1} (-1)^i r_i f_i.$$ 

The $r_i$ are needed to shift the domain of definition for the restricted map $f_i$ to the standard $n-1$ simplex, which is the source for the $n-1$ chains. A check shows that $\partial^2 = 0$, and singular homology is defined as the homology of the complex $\mathcal{S}$.

Theorem 2.11. If $X$ and $Y$ are homotopic, then $H_i(X) \cong H_i(Y)$.

Proof. Since $X$ and $Y$ are homotopic, by Definition 1.9 of Chapter 4 there exist continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ satisfying $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. These maps induce an isomorphism on the chains, and hence on homology. \hfill \square

We remarked earlier that it is not obvious that simplicial homology is a topological invariant. In fact, there is an isomorphism between the singular and simplicial homology of a space $X$, and we close with a sketch of the proof.

Theorem 2.12. For a topological space $X$ which can be triangulated, there is an isomorphism from the simplicial homology of $X$ to the singular homology of $X$.

Proof. Let $K$ be a triangulation of $X$, so $S_i(X) = S_i(K)$. Then the map

$$C_i(K, R) \xrightarrow{\phi} S_i(X)$$

defined by sending an $i$-simplex $\sigma \in K_i$ to the map taking the standard $i$-simplex to $\sigma$ commutes with the differentials of $C_\bullet$ and $S_\bullet$, so by Lemma 1.4 of Chapter 6 there is an induced map

$$H_i(K, R) \xrightarrow{\phi_*} H_i(X)$$

To show the map is an isomorphism, first prove it when $K$ is a finite simplicial complex, by inducting on the number of simplices. When $K$ is infinite, a class in singular homology is supported on a compact subset $C$. Then one shows that there is a finite subcomplex of $K$ containing $C$, reducing back to the finite case. \hfill \square
3. CW Complexes and Cellular Homology

CW complexes are less intuitive than simplicial or singular complexes. The utility of CW complexes is that they provide a more economical way to construct spaces than can be done when using other methods, and this manifests in simpler homology computations. A topological space is Hausdorff if for any pair of points \( p_1 \neq p_2 \) there exist open neighborhoods \( N_{\delta_{p_1}}(p_1) \) such that

\[
N_{\delta_{p_1}}(p_1) \cap N_{\delta_{p_2}}(p_2) = \emptyset.
\]

A CW complex is a Hausdorff topological space built inductively by attaching open balls \( B^n \) to a lower dimensional scaffolding \( X^{n-1} \) in a prescribed way. A zero dimensional CW complex \( X^0 \) is simply a set of points, and a one-dimensional CW complex is built by gluing copies of the unit interval \([0, 1]\) to \( X^0 \), with the requirement that \( \partial [0, 1] = \{0, 1\} \mapsto X^0 \). In general, we have

**Definition 3.1.** An \( n \)-dimensional CW complex is the union of

- Open balls \( B^n \), known as **cells**.
- An \( n - 1 \) dimensional CW complex \( X^{n-1} \), the \( n - 1 \) **skeleton**,

quotiented by the image of continuous attaching maps for each ball \( B^n \):

\[
\partial (B^n) = S^{n-1} \mapsto X^{n-1}
\]

**Example 3.2.** The \( n \)-sphere \( S^n \) may be constructed with only two cells: an \( n \)-cell \( B^n \) and zero cell \( p \), with gluing map collapsing \( \partial (B^n) = S^{n-1} \mapsto X^{n-1} \) to the point \( p \). The **algebraic** chain complex we construct has nonzero terms only in positions \( n \) and \( 0 \), and all differentials are zero, allowing computation of the homology of \( S^n \) with no effort at all. Other examples of CW complexes:

- Simplicial and Polyhedral Complexes.
- Projective Space and Grassmannians.
- Compact Surfaces.
- Smooth Manifolds*

Of course, the classes above overlap; the asterisk for manifolds reflects the fact that there is a CW complex which is homotopic to a given manifold.

CW complexes were introduced by J.H.C. Whitehead; the nomenclature arises from the fact that a CW complex \( X \)

- is closure finite (C): the closure of a cell of \( X \) intersects only a finite number of other cells of \( X \).
- has the weak topology (W): \( Y \subseteq X \) is closed iff \( Y \cap e \) is closed for every cell \( e \) of \( X \).
To define homology for CW complexes, we need relative homology, which appears in §2.2 of the next chapter. Here is a sneak preview: suppose \( X \subseteq Y \) are topological spaces; for simplicity let \( X \) be a simplicial complex which is a subcomplex of \( Y \). Then for each \( k \) we have short exact sequences
\[
0 \to C_k(X) \xrightarrow{i} C_k(Y) \xrightarrow{j} C_k(Y)/C_k(X) \to 0.
\]
The maps \( i \) and \( j \) commute with the boundary operators, and Theorem 1.6 of Chapter 6 shows this yields a long exact sequence
\[
\cdots \to H_k(X) \xrightarrow{i_*} H_k(Y) \xrightarrow{j_*} H_k(Y,X) \xrightarrow{\delta} H_{k-1}(X) \to \cdots
\]
Here \( H_k(Y,X) \) denotes the homology of the quotient complex whose \( k^{th} \) term is \( C_k(Y)/C_k(X) \). We apply this to construct a chain complex from a CW complex \( X \), using the \( k \)-skeleta \( X^k \) of \( X \) as building blocks.

**Lemma 3.3.** For a CW complex \( X \), define \( D_p(X) = H_p(X^p, X^{p-1}) \), and define
\[
D_p(X) \xrightarrow{\partial} D_{p-1}(X)
\]
via the composite map \( \partial = j_*\delta \)
\[
\begin{array}{ccc}
H_p(X^p, X^{p-1}) & \xrightarrow{\partial} & H_{p-1}(X^{p-1}, X^{p-2}) \\
\downarrow{\delta} & & \downarrow{j_*} \\
H_{p-1}(X^{p-1})
\end{array}
\]
Then \( \partial^2 = 0 \).

**Proof.** Because \( \partial^2 = (j_*\delta) \cdot (j_*\delta) \), the result follows from
\[
\delta j_* = 0,
\]
which is a consequence of the exactness of the sequence in Equation 3.1. Notice that since \( X^k \) is dimension \( k \), \( H_n(X^k) = 0 \) for \( n > k \). \( \square \)

**Example 3.4.** The torus \( T^2 \) and Klein bottle \( KB \) can both be constructed as a CW complex with a single two cell, a pair of one cells, and a single zero cell, as below

\[
\begin{array}{c}
\begin{array}{c}
T^2 \\
\varepsilon_1 \\
\varepsilon_2
\end{array} & \xrightarrow{e_1} & \xrightarrow{e_2} \\
\varepsilon_1 \\
\varepsilon_2
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
KB \\
\varepsilon_1 \\
\varepsilon_2
\end{array} & \xrightarrow{e_1} & \xrightarrow{e_2} \\
\varepsilon_1 \\
\varepsilon_2
\end{array}
\]

Hence the cellular homology with \( \mathbb{Z} \)-coefficients is given for both spaces by
\[
0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z} \to 0.
\]
All differentials are zero, except $\partial_2$ of the Klein bottle, which sends the two cell to twice the vertical edge. So

$$
\begin{array}{c|c}
T^2 & KB \\
H_0 & \mathbb{Z} \\
H_1 & \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \\
H_2 & 0 \\
\end{array}
$$

4. Poincaré and Alexander Duality

Henri Poincaré discovered his duality theorem in 1893; it has an intrinsic appeal because it concerns symmetry. In Poincaré duality, there is a single underlying object (the manifold $M$), and the duality involves the homology of $M$. James Alexander proved his duality theorem in 1915; in contrast to Poincaré duality, Alexander duality concerns two different spaces. We begin with a statement of the theorems. For simplicity, we use coefficients in a field $R$.

4.1. Statement of theorems and examples.

**Theorem 4.1.** \textbf{[Poincaré Duality]} Let $M$ be a connected compact orientable (boundaryless) manifold of dimension $n$. Then there is a perfect pairing

$$H_k(M, R) \otimes H_{n-k}(M, R) \rightarrow R$$

This says that $\text{Hom}_R(H_{n-k}(M), R) \simeq H_k(M, R)$, which is often phrased in terms of cohomology

$$H^k(M, R) \simeq H_{n-k}(M, R)$$

Cohomology is described in Chapter 6; here’s a quick preview: for a simplicial complex $\Delta$, define $C^i(\Delta, R) = \text{Hom}_R(C_i(\Delta, R))$; it is easily checked that the $C^i$ form a complex $\mathcal{C}$; $H^i(\mathcal{C})$ is the cohomology at the $i^{th}$ position. When $R = \mathbb{Z}$ the rank $b_i$ of $H_i(\Delta, \mathbb{Z})$ is called the $i^{th}$ Betti number of $\Delta$, so for $R$ a field it is simply dimension. Poincaré duality asserts that for $M$ as in the statement of Theorem 4.1, the vector $b(M) = (b_0, \ldots, b_n)$ of Betti numbers is a palindrome: $(b_0, \ldots, b_n) = (b_n, \ldots, b_0)$.

**Example 4.2.** Our computations earlier in the chapter show that

$$
\begin{align*}
\text{b}(S^1) &= (1, 1) & \text{by Example 2.5.} \\
\text{b}(S^2) &= (1, 0, 1) & \text{by Example 2.8.} \\
\text{b}(S^n) &= (1, 0^{n-1}, 1) & \text{by Example 3.2.}
\end{align*}
$$

Example 3.4 showed that the Torus has Betti vector $(1, 2, 1)$, whereas the Klein Bottle has Betti vector $(1, 1, 0)$. We don’t expect Poincaré duality to hold for the Klein Bottle, since it is not orientable.
Theorem 4.3. [Alexander Duality] For a compact and locally contractible $X \subseteq S^n$, the homology of $X$ and of $S^n \setminus X$ are related via $$\tilde{H}_k(S^n \setminus X) \simeq \tilde{H}^{n-1-k}(X)$$ where $\tilde{H}_i$ denotes reduced cohomology.

In terms of the reduced Betti vector, this says that $\tilde{b}(S^n \setminus X)$ is obtained by reversing $\tilde{b}(X)$, then shifting left by one.

Example 4.4. Let $n = 2$, with $S^2$ triangulated as in $\Delta_1$ of Example 2.8, and let $X$ be the subcomplex with maximal faces $\{[1, 2], [2, 3], [3, 1]\}$, so $X$ is homotopic to $S^1$. The homology of $X$ was computed in Example 2.5. Appending zeros so the reduced Betti vector of $X$ has length $n + 1 = 3$ yields $$\tilde{b}(X) = (0, 1, 0).$$ What about $S^2 \setminus X$? Since singular and simplicial homology agree, we may use singular homology to compute. Since singular homology is homotopy invariant, we see that removing the $X = S^1$ “belt” from $S^2$ results in two disconnected disks, which are each homotopic to a point. Therefore $b(S^2 \setminus X) = (2, 0, 0)$, hence $$\tilde{b}(S^2 \setminus X) = (1, 0, 0)$$ reverse $(1, 0, 0)$ to obtain $(0, 0, 1)$ shift $(0, 0, 1)$ left one step $(0, 1, 0) = \tilde{b}(X)$.

4.2. Alexander Duality: Proof and TDA Application. We will prove a combinatorial version of Alexander Duality, following the elegant argument of Björner-Tancer [12]. The setup is the following: $\Delta$ will be a simplicial complex on vertex set $V = \{v_1, \ldots, v_n\}$. We write $S$ for the simplicial complex on $V$ whose maximal faces correspond to the $n$ subsets of $V$ having $n - 1$ vertices; clearly $S$ is homotopic to $S^{n-2}$.

Definition 4.5. The combinatorial Alexander dual of $\Delta$ is $$\Delta^* = \{\sigma \subset V \mid \sigma \not\in \Delta\}.$$ The condition that $\sigma \not\in \Delta$ means that $\sigma$ is the complement of a non-face of $\Delta$.

Example 4.6. For $X$ in Example 4.4, the nonfaces of $\Delta$ are $$ed\{[1, 2, 3], [1, 4], [2, 4], [3, 4], [4]\},$$ so the complements of the nonfaces are $$\{[4], [2, 3], [1, 3], [1, 2], [1, 2, 3]\}.$$ Hence the maximal faces of $\Delta^*$ are the vertex $[4]$ and the triangle $[1, 2, 3]$, so $\Delta^*$ is homotopic to a pair of points.
Since $|V| = n$, the corresponding sphere is an $S^{n-2}$, so combinatorial Alexander duality will take the form

$$\tilde{H}_i(\Delta) \simeq \tilde{H}^{n-3-i}(\Delta^*).$$

**Lemma 4.7.** For $j \in \sigma \in \Delta$, $\text{sgn}(j, \sigma) = (-1)^{i-1}$, where $j$ is the $i^{th}$ smallest element in the set $\sigma$. Let

$$p(\sigma) = \prod_{i \in \sigma} (-1)^{i-1}.$$

Then for $k \in \sigma$,

$$\text{sgn}(k, \sigma)p(\sigma \setminus k) = \text{sgn}(k, \sigma \cup k)p(\sigma).$$

**Exercise 4.8.** Show both sides of the expression above equal $(-1)^{k-1}$. \(\Diamond\)

Let $e_\sigma \in \tilde{C}_i$ denote the element corresponding to $\sigma \in \Delta_i$, so that

$$\partial_i(e_\sigma) = \sum_{j \in \sigma} \text{sgn}(j, \sigma)e_{\sigma \setminus j}.$$

Dualizing to the $C_i$, this means the dual maps in cohomology are given by

$$\partial^i(e^*_\sigma) = \sum_{\substack{j \notin \sigma \\text{ or } \sigma \cup j \in \Delta}} \text{sgn}(j, \sigma \cup j)e^*_{\sigma \cup j}.$$

Equation 4.1 will follow from the following two isomorphisms, where $2^V$ is the full (solid) $n - 1$ simplex $[v_1, \ldots, v_n]$.

$$\tilde{H}_i(\Delta) \simeq \tilde{H}_{i+1}(2^V, \Delta)$$

and

$$\tilde{H}_{i+1}(2^V, \Delta) \simeq \tilde{H}^{n-3-i}(\Delta^*)$$

Equation 4.4 follows from the long exact sequence in relative homology for the pair $\Delta \subseteq 2^V$, combined with the vanishing of all $\tilde{H}_i(2^V)$. The vanishing of $\tilde{H}_i(2^V)$ for all $i$ is immediate, since $2^V$ is homotopic to a point. So the key is to show that Equation 4.5 holds.

**Proof.** (Combinatorial Alexander Duality–Equation 4.1) Write the differential in relative homology for the pair $(2^V, \Delta)$ as $d_j$ and the relative chains as $R_j$; the operator $d_j$ is given by essentially the same formula as Equation 4.2, except that chains with image in $\tilde{C}_i(\Delta)$ are zero; accounting for this we have:

$$d_j(e_\sigma) = \sum_{\substack{k \in \sigma \\text{ or } \sigma \setminus k \in \Delta}} \text{sgn}(k, \sigma)e_{\sigma \setminus k}.$$
Similarly, writing the relative cochains as $C_j$ and the relative coboundary operator as $d_j^j$, we have

\begin{equation}
    d_j^j(e_\sigma^*) = \sum_{k \notin \sigma, k \in \Delta} \text{sgn}(k, \sigma \cup k)e_{\sigma \cup k} = \sum_{k \in \sigma, k \notin \Delta} \text{sgn}(k, \sigma \cup k)e_{\sigma \cup k}.
\end{equation}

To complete the proof we bring the operator $p$ from Lemma 4.7 into play. Define

$$R_j \phi_j \to C^{n-j-2}$$

via

$$\phi_j(e_\sigma) = p(\sigma)e_\sigma^*.$$

Then the diagram

$$\cdots \to R_j \xrightarrow{d_j} R_{j-1} \to \cdots$$

commutes, and therefore induces the isomorphism of Equation 4.1. \hfill \Box

**Exercise 4.9.** Check that indeed $\phi_{j-1}d_j = d^{n-1-j}\phi_j$. \hfill \diamond

We close this section with a discussion of the use of Alexander Duality by Adams-Carlsson in [1] on a problem in sensor networks. Let $D \subseteq \mathbb{R}^d$ be a bounded domain homeomorphic to a ball, and suppose we have a set of sensors $S = \{v_1, \ldots, v_n\}$, each providing coverage of a unit $d$-ball, wandering through $D$. The sensors do not know coordinate information, but do know when their coverage region overlaps with another sensor. Define a connectivity graph, whose vertices are sensors, with an edge connecting sensors if their coverage overlaps. We want to detect intruders during a time interval $t \in I = [0, 1]$. Let

$$X(t) = \bigcup_{v_i(t) \in S} B_{v_i(t)}$$

and $X = \bigcup_{t \in I} X(t) \times \{t\} \subseteq D \times I$.

So $X(t)$ is the coverage of the sensor network at time $t$, and $X$ is the subset of spacetime covered by sensors.

**Definition 4.10.** An evasion path is a section of the projection map

$$X^c = D \times I \setminus X \xrightarrow{\pi} I.$$ 

Hence an evasion path is a map $I \xrightarrow{s} X^c$ such that $s(t) \notin X(t)$ for any $t \in I$.

The results on this problem are all in terms of various flavors of persistent homology, which is the topic of Chapter 7. In [36], Ghrist-de Silva use the Rips complex to give a necessary homological condition for existence of an evasion path. Building on this, Adams-Carlsson combine zigzag persistent homology with Alexander duality to prove that if an evasion path exists, then there is a full length interval in the zigzag barcode for the $d-1^{st}$ homology of the (stacked) Čech complex.
4.3. Poincaré duality. Poincaré duality is in some sense the paradigm for duality theories in topology. It can be proved in a purely algebraic manner, via cap product with the fundamental class. A more easily visualized low-tech proof involves several fundamental tools such as barycentric subdivision and dual cell decomposition, and this is the proof we sketch below.

**Definition 4.11.** Let $\sigma_n$ be a geometric $n$-simplex, so that $\sigma_n$ is the convex hull of vertices $\{v_0, \ldots, v_n\}$. The barycenter $b(\sigma_n)$ of $\sigma_n$ is

$$b(\sigma_n) = \frac{1}{n+1} \sum_{i=0}^{n} v_i.$$ 

The first barycentric subdivision $B(\sigma_n)$ of $\sigma_n$ is obtained iteratively:
- $B_1$ is the the one-dimensional simplicial complex obtained by barycentrically subdividing the edges of $\sigma_n$.
- $B_{i+1}$ is the subdivision of the $i+1$ skeleton of $\sigma_n$ obtained by coning the barycenters of the $i+1$ simplices with $B_i$.

**Example 4.12.** The barycentric subdivision of a 2-simplex:

![Barycentric Subdivision](image)

**Exercise 4.13.** [Poincaré-Hopf] A beautiful application of barycentric subdivision involves the Poincaré-Hopf theorem; we follow the treatment in §8 of [52]. Let $\mathcal{V}$ be a vector field on a smooth, compact orientable surface $S$ of genus $g$—"a sphere with $g$ handles". A point $p \in S$ is a singularity of $\mathcal{V}$ if $\mathcal{V}(p) = 0$ or is undefined; the index $\text{ind}_p(\mathcal{V})$ is defined to be the integral of $\mathcal{V}$ around a small counterclockwise loop centered at $p$, as in vector calculus. The Poincaré-Hopf theorem is

$$\sum_{p \in \text{sing}(\mathcal{V})} \text{ind}_p(\mathcal{V}) = \chi(S).$$ 

This is amazing: a property of vector fields is governed by topology. Prove Equation 4.8 for the following special class of vector fields.
Let $\Delta$ be a triangulation of $S$, and construct a vector field $\mathcal{V}$ on $S$ as “flow towards the barycenters of the faces of $\Delta$”, depicted below for a two simplex:

Since the barycenter of a vertex $\nu$ is $\nu$ itself, the singularities of $\mathcal{V}$ occur only at the barycenters of the faces of $\Delta$. The index computation is local; translate so $p$ is the origin in $\mathbb{R}^2$ and use vector calculus to compute that

$$\text{ind}_p(\mathcal{V}) = 1 \quad \text{if } p = b(\nu) \text{ for } \nu \text{ a vertex of } \Delta.$$  

Hint: $\mathcal{V} = \langle x, y \rangle$.

$$\text{ind}_p(\mathcal{V}) = -1 \quad \text{if } p = b(\epsilon) \text{ for } \epsilon \text{ an edge of } \Delta.$$  

Hint: $\mathcal{V} = \langle y, x \rangle$.

$$\text{ind}_p(\mathcal{V}) = 1 \quad \text{if } p = b(\tau) \text{ for } \tau \text{ a triangle of } \Delta.$$  

Hint: $\mathcal{V} = \langle -x, -y \rangle$.

Why does the result follow from this? Example 2.4 will be useful.

With barycentric subdivision in hand, we next describe the dual cell subdivision $D$ of a simplicial complex $\Delta$. The resulting object $D$ is a CW complex.

**Definition 4.14.** Let $\Delta$ be an $n$-dimensional geometric simplicial complex, and let $\tau$ be a $k$-face of $\Delta$. We define an $n-k$ cell $\tau^*$ dual to $\tau$ as follows.

- Form the first barycentric subdivision of $\Delta$: the union of the first barycentric subdivision of each maximal face $\sigma$ of $\Delta$. Note that if $\Delta$ triangulates a manifold, the maximal faces of $\Delta$ are all equidimensional.

- For each maximal face $\sigma$ containing $\tau$ and intermediate face $\tau'$ such that $\tau \subseteq \tau' \subseteq \sigma$, let $b_{\tau'}$ be the barycenter of $\tau'$. The dual cell is defined by letting $\tau_{\sigma} = \tau^* \cap \sigma = \text{Conv}\{b_{\tau'} \mid \tau \subseteq \tau' \subseteq \sigma\}$, and

$$\tau^* = \bigcup_{\tau \subseteq \sigma \in \Delta_n} \tau_{\sigma}.$$

Notice that $\tau^* \cap \tau$ meet in a unique point $b_\tau$.

Consider two homology classes $C$ and $C'$ of complementary dimensions $k$ and $n-k$ on an $n$-dimensional manifold. Cycles are equivalence classes, so they can be moved. One way to formulate Poincaré duality is that if cycles are moved so that $C$ and $C'$ meet transversely at a point $p$, then

$$T_p(X) = T_p(C) \oplus T_p(C').$$

See [59] for a proof along these lines. This is also essentially what happens with the dual cell decomposition.
Example 4.15. For the $S^2$ with triangulation as in Example 2.8, the dual cell decomposition is

For an edge $\tau$ with vertices $\{i, j\}$, the barycenter $b_\tau$ is labelled $ij$, and similarly for triangles.

The dual cell decomposition gives a $1 - 1$ map

$$C^k(\Delta) \rightarrow C_{n-k}(D) = D_{n-k}.$$  

The relation between the coboundary map $C^{k-1}(\Delta) \rightarrow C^k(\Delta)$ and the boundary map of the dual cell complex $D_{n-k} = C_{n-k}(D) \rightarrow C_{n-k-1}(D) = D_{n-k-1}$ is illustrated in Example 4.15: the boundary relations on the triangulation $\Delta$ correspond to the incidence relations on the dual cell decomposition $D$. The difficult part of Poincaré duality is getting the signs correct in the diagram

$$\cdots \rightarrow C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{\phi_k} \cdots$$

$$\cdots \rightarrow D_{n-k+1} \xrightarrow{d_{n-k+1}} D_{n-k} \xrightarrow{\phi_k} \cdots$$

The construction of $\phi$ runs as follows. First, orient the top dimensional simplices of $\Delta$ so that $\omega = \sum_{\sigma_i \in \Delta_n} \sigma_i$ satisfies $\partial_n(\omega) = 0$. This is possible since $\Delta$ triangulates a manifold $M$, and $\partial(M) = 0$. A basis for $C_n(\Delta)$ is given by the $\sigma \in \Delta_n$, so a basis for $C_n(\Delta)$ is given by the $\sigma^*$. The dual block to $\sigma$ is the zero-cell $\sigma^*$ corresponding to $b(\sigma)$, so we have a map $\phi_n : C^n \rightarrow D_0$ which sends $\sigma^*$ to $\sigma^*$. To define $\phi_{n-1}$, let $\xi$ be an oriented $n - 1$ face of $\Delta$. We require

$$d_1 \cdot \phi_{n-1}(\xi^*) = \phi_n d^{n-1}(\xi^*).$$

There exist a pair of faces $\sigma_i \in \Delta_n$ so that $\xi = \sigma_1 \cap \sigma_2$. Since $\partial_n(\omega) = 0$, $\xi$ appears with opposite signs in the expressions for $\partial(\sigma_1)$ and $\partial(\sigma_2)$, so we can assume

$$d^{n-1}(\xi^*) = \sigma_1^* - \sigma_2^*,$$

which implies $\phi_n \cdot d^{n-1}(\xi^*) = \sigma_1^* - \sigma_2^*$. Define $\phi_{n-1}(\xi^*) = [\sigma_2^*, \xi^*] + [\xi^*, \sigma_1^*]$. Then we have

$$d_1(\phi_{n-1}(\xi^*)) = d_1([\sigma_2^*, \xi^*]) + d_1([\xi^*, \sigma_1^*]) = \sigma_1^* - \sigma_2^*,$$

as required.
The general recipe for $\phi$ is as follows: let $\tau$ be an element of $C_k$, and $\tau^\vee$ the corresponding element of $C^k$. Then the corresponding element of $D_{n-k}$ is obtained as

$$\phi_k(\tau^\vee) = \sum_{\sigma \in \Delta_n \atop \tau \in \Delta_k \atop \tau \subset \sigma} \text{sgn}(\tau, \sigma)\tau_\sigma,$$

where $\text{sgn}(\tau, \sigma)$ is the sign of $\tau \in \partial(\sigma)$.

A computation shows that

$$\phi_k \cdot d^{k-1} = d_{n-k+1} \cdot \phi_{k-1}.$$ 

This concludes our sketch of the proof of Poincaré duality. When $M$ is compact but not orientable, Theorem 4.1 holds, with $\mathbb{Z}/2$ coefficients. In this case, we don’t need to worry about $\phi$, because $1 = -1$, which simplifies the proof.

**Example 4.16.** The differential for the Klein bottle was computed in Example 3.4. Let $\sigma$ be the two-cell, and $\tau_1, \tau_2$ the vertical and horizontal edges.

$$\partial_2(\sigma) = 2 \cdot \tau_2,$$

With $\mathbb{Z}/2$ coefficients, we see that the homology of the Klein bottle is the same as that of the torus, so

$$b_{\mathbb{Z}/2}(\text{Klein Bottle}) = (1, 2, 1)$$

and Poincaré duality holds.
Chapter 6

Homology II: Algebraic Tools and Applications

Homological algebra revolves around the study of complexes. Proofs often involve comparing different paths between objects, and for this reason it is sometimes referred to as “diagram chasing”. Many of the constructions in homological algebra have their origins in algebraic topology, so it is not surprising that the tools and techniques of homological algebra play a key role in investigating topological objects. Homological algebra is closely connected to abstract category theory, and can become very abstruse. For this reason, we confine ourselves to the setting of complexes of modules over a Noetherian ring (equivalently, coherent sheaves on a Noetherian space). This allows us to illustrate the key aspects of the theory without getting bogged down in abstraction. We start by recalling Definition 2.5: a complex of $\mathbb{R}$-modules is a sequence of modules and homomorphisms

$$
\cdots M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2} \cdots
$$

such that $\text{im}(d_i) \subseteq \ker(d_{i-1})$. In this chapter, we tackle

- Fundamentals: Snake Lemma, Long Exact Sequence in Homology.
- Applications and Examples: Mayer-Vietoris, Relative Homology.
- Cohomology: Simplicial, Čech, de Rham theories.
- Ranking and Hodge Theory.

The last of these topics, Ranking and Hodge Theory, draws on work of Jiang-Lim-Yao-Yuan in [66], which applies a central tool of algebraic geometry–Hodge Theory–to the Netflix Problem. The goal is to produce a coherent ranking from partial and inconsistent survey results, such as viewer ranking of movies. In particular, some rankings may involve loops: $a > b > c > a$. 

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1. Fundamentals: Snake Lemma, Long Exact Sequence

This section lays the groundwork for the rest of the chapter. The centerpiece is the existence of a long exact sequence in homology arising from a short exact sequence of complexes, and the key ingredient in proving this is the Snake Lemma.


**Lemma 1.1** (The Snake Lemma). For a commutative diagram of $R$–modules with exact rows

\[
\begin{array}{ccc}
A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \rightarrow 0 \\
\downarrow{f_1} & & \downarrow{f_2} & & \downarrow{f_3} \\
0 & \xrightarrow{b_1} & B_1 & \xrightarrow{b_2} & B_2 & \xrightarrow{b_3} B_3
\end{array}
\]

there is an exact sequence:

\[
\begin{array}{c}
\ker f_1 \rightarrow \ker f_2 \rightarrow \ker f_3 \xrightarrow{\delta} \\
\coker f_1 \rightarrow \coker f_2 \rightarrow \coker f_3
\end{array}
\]

The name comes from the connecting homomorphism $\delta$, which *snakes* from the end of the top row to the start of the bottom row.

**Proof.** The key to the snake lemma is defining the connecting homomorphism $\delta : \ker(f_3) \rightarrow \coker(f_1)$. Let $a \in \ker(f_3)$. Since the map $a_2$ is surjective, there exists $b \in A_2$ such that $a_2(b) = a$, hence

\[f_3(a_2(b)) = f_3(a) = 0.\]

Since the diagram commutes,

\[0 = f_3(a_2(b)) = b_2(f_2(a)).\]

By exactness of the rows, $f_2(a) \in \ker(b_2) = \text{im}(b_1)$. So there exists $c \in B_1$ with

\[b_1(c) = f_2(a).\]

Let $\tilde{c}$ be the image of $c$ in $\coker(f_1) = B_1/\text{im}(f_1)$, and define

\[\delta(a) = \tilde{c}.\]

To show that $\delta$ is well defined, consider where choices were made: if $a_2(b') = a$, then $a_2(b - b') = 0$, hence $b - b' \in \ker(a_2) = \text{im}(a_1)$, so there exists $e \in A_1$ with $a_1(e) = b - b'$, hence $b = b' + a_1(e)$. Applying $f_2$ we have $f_2(b) = f_2(b') + f_2(a_1(e)) = f_2(b') + b_1(f_1(e))$. But $b_1$ is an inclusion, so $f_2(b)$ and $f_2(b')$ agree modulo $\text{im}(f_1)$, so $\delta$ is indeed a well defined map to $\coker(f_1)$.

**Exercise 1.2.** Check that the Snake is exact.  \(\diamond\)
Definition 1.3. If $A_\bullet$ and $B_\bullet$ are complexes, then a **morphism of complexes** $\phi$ is a family of homomorphisms $A_i \xrightarrow{\phi_i} B_i$ making the diagram below commute:

![Diagram of morphism of complexes](image)

Lemma 1.4 (Induced Map on Homology). A morphism of complexes induces a map on homology.

**Proof.** To show that $\phi_i$ induces a map $H_i(A) \rightarrow H_i(B)$, take $a_i \in A_i$ with $\partial_i(a_i) = 0$. Since the diagram commutes,

$$0 = \phi_{i-1}\partial_i(a_i) = \delta_i\phi_i(a_i)$$

Hence, $\phi_i(a_i)$ is in the kernel of $\delta_i$, so we obtain a map $\ker \partial_i \rightarrow H_i(B)$. If $a_i = \partial_{i+1}(a_{i+1})$, then

$$\phi_i(a_i) = \phi_i\partial_{i+1}(a_{i+1}) = \delta_{i+1}\phi_{i+1}(a_{i+1}),$$

so $\phi$ takes the image of $\partial$ to the image of $\delta$, yielding a map $H_i(A) \rightarrow H_i(B)$. □

Definition 1.5. A **short exact sequence of complexes** is a commuting diagram:

![Diagram of short exact sequence](image)

where the columns are exact and the rows are complexes.

We now come to the fundamental result of this section.

Theorem 1.6. [Long Exact Sequence in Homology] A short exact sequence of complexes yields a long exact sequence in homology:

$$\cdots \rightarrow H_{n+1}(C) \rightarrow H_n(A) \rightarrow H_n(B) \rightarrow H_n(C) \rightarrow H_{n-1}(A) \rightarrow \cdots$$
Proof. We induct on the length of the complexes. When the diagram above has only two columns, we are in the setting of the Snake Lemma, yielding the base case. So now suppose the result holds for complexes of length \( n - 1 \), and consider a complex of length \( n \). Prune the two leftmost columns off the short exact sequence of complexes in Definition 1.5

\[
\begin{array}{c}
0 & \rightarrow & A_n & \xrightarrow{d_n} & A_{n-1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & B_n & \xrightarrow{d_n} & B_{n-1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C_n & \xrightarrow{d_n} & C_{n-1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 \\
\end{array}
\]

By the Snake Lemma, we have

\[
0 \to H_n(A) \to H_n(B) \to H_n(C) \to \text{coker}(d_n^a) \to \text{coker}(d_n^b) \to \text{coker}(d_n^c) \to 0
\]

A diagram chase as in the proof of the snake lemma shows that

\[
\text{im}(\delta) \subseteq \text{ker}(d_{n-1}^a)/\text{im}(d_n^a) = H_{n-1}(A).
\]

The pruned complex of length \( n - 1 \) begins as

\[
\begin{array}{c}
0 & \rightarrow & A_{n-1}/(d_n^a + \text{im}(H_n(C))) & \rightarrow & A_{n-2} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & B_{n-1}/d_n^b & \rightarrow & B_{n-2} & \rightarrow & \cdots 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & C_{n-1}/d_n^c & \rightarrow & C_{n-2} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 \\
\end{array}
\]

By the Induction Hypothesis, we have a long exact sequence

\[
(1.1) \quad 0 \to H_{n-1}(A)/\text{im}(H_n(C)) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots
\]
Since \( \text{im}(\delta) \subseteq H_{n-1}(A) \), we also have an exact sequence

\[
\cdots \rightarrow H_n(C) \xrightarrow{\delta} H_{n-1}(A) \rightarrow H_{n-1}(A)/H_n(C) \rightarrow 0.
\]  

To conclude, splice the surjection of Equation 1.2

\[ H_{n-1}(A) \rightarrow H_{n-1}(A)/H_n(C) \rightarrow 0 \]

with the inclusion of Equation 1.1

\[ 0 \rightarrow H_{n-1}(A)/H_n(C) \rightarrow H_{n-1}(B), \]

to obtain a map \( H_{n-1}(A) \rightarrow H_{n-1}(B) \), which completes the induction. \( \square \)

1.2. Chain Homotopy. When do two morphisms of complexes induce the same map on homology?

**Definition 1.7.** If \( A \) and \( B \) are complexes, and \( \alpha, \beta \) are morphisms of complexes, then \( \alpha \) and \( \beta \) are **homotopic** if there exists a family of homomorphisms \( A_i \xrightarrow{\gamma_i} B_{i+1} \) such that for all \( i \), \( \alpha_i - \beta_i = \delta_i + 1 \gamma_i + \gamma_i - 1 \partial_i \). Notice that \( \gamma \) need not commute with \( \partial \) and \( \delta \).

**Theorem 1.8.** Homotopic maps induce the same map on homology.

**Proof.** It suffices to show that if \( \alpha_i = \delta_{i+1} \gamma_i + \gamma_{i-1} \partial_i \) then \( \alpha \) induces the zero map on homology. But if \( a_i \in H_i(A) \), then since \( \partial_i(a_i) = 0 \),

\[
\alpha_i(a_i) = \delta_{i+1} \gamma_i(a_i) + \gamma_i - 1 \partial_i(a_i) = \delta_i + 1 \gamma_i(a_i) \in \text{im}(\delta),
\]

so \( \alpha_i(a_i) = 0 \) in \( H_i(B) \). \( \square \)

2. Applications and Examples: Mayer-Vietoris, Relative Homology

As we noted at the beginning of the chapter, homological algebra grew out of topology, and we illustrate this now.

2.1. Mayer-Vietoris sequence. A central idea in mathematics is to study an object by splitting it into simpler component parts. The Mayer-Vietoris sequence provides a way to do this in the topological setting. Basically, Mayer-Vietoris is a “divide and conquer” strategy. Given a topological space or simplicial complex \( Z \), if we write

\[
Z = X \cup Y
\]

then there will be relations between \( X, Y, Z \) and the intersection \( X \cap Y \).

**Theorem 2.1** (Mayer-Vietoris). Let \( X \) and \( Y \) be simplicial complexes, with \( Z = X \cup Y \). Then there is a short exact sequence of complexes

\[
0 \rightarrow C_\bullet(X \cap Y) \rightarrow C_\bullet(X) \oplus C_\bullet(Y) \rightarrow C_\bullet(X \cup Y) \rightarrow 0,
\]

yielding a long exact sequence in homology.
**Proof.** We need to show that for a fixed index $i$, there are short exact sequences

$$0 \to C_i(X \cap Y) \to C_i(X) \oplus C_i(Y) \to C_i(X \cup Y) \to 0$$

Define the map from

$$C_i(X) \oplus C_i(Y) \to C_i(X \cup Y) \text{ via } (\sigma_1, \sigma_2) \mapsto \sigma_1 - \sigma_2$$

The kernel of the map is exactly $C_i(X \cap Y)$. \hfill \Box

**Example 2.2.** In Example 2.8 of Chapter 5, the complex $\Delta_2$ is constructed from a pair of hollow tetrahedra by dropping a triangle from each one, then identifying the result along the edges of the deleted triangle. From a topological perspective, this is equivalent to gluing two disks along their boundaries. The top dimensional faces of the complexes involved are

$$X = \{[1, 2, 4], [1, 3, 4], [2, 3, 4]\}$$
$$Y = \{[1, 2, 5], [1, 3, 5], [2, 3, 5]\}$$
$$X \cap Y = \{[1, 2], [1, 3], [2, 3]\}$$
$$X \cup Y = \{[1, 2, 4], [1, 3, 4], [2, 3, 4], [1, 2, 5], [1, 3, 5], [2, 3, 5]\}$$

So our short exact sequence of complexes, with $R$ coefficients, has the form

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
X \cap Y : & 0 & \to & R^3 \\
& \downarrow & \uparrow & \partial_C^1 \\
& \to & R^3 & \to & 0 \\
X \oplus Y : & 0 & \to & R^6 \\
& \downarrow & \uparrow & \partial_B^2 \\
& & R^6 & \to & R^{12} \\
& & \downarrow & \uparrow & \partial_B^1 \\
& & & R^{12} & \to & R^8 \\
& & & \downarrow & \uparrow & \partial_C^1 \\
& & & & R^8 & \to & 0 \\
X \cup Y : & 0 & \to & R^6 \\
& \downarrow & \uparrow & \partial_C^2 \\
& & R^6 & \to & R^9 \\
& & \downarrow & \uparrow & \partial_C^1 \\
& & & R^9 & \to & R^5 \\
& & & \downarrow & \uparrow & \partial_C^1 \\
& & & & R^5 & \to & 0
\end{array}
\]

We have written the $\partial_i$ with superscripts to distinguish them. With oriented basis for $C_1(X \cap Y)$ as above and basis for $C_0(X \cap Y) = \{[1], [2], [3]\}$ we have

$$\partial_C^1 = \begin{bmatrix}
-1 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 1
\end{bmatrix}$$

So we have

$$H_1(X \cap Y) \simeq R^1 \simeq H_0(X \cap Y).$$

An easy check shows that for both $X$ and $Y$, $H_i = 0$ if $i \neq 0$, so from the long exact sequence in homology we have

$$H_2(X \cup Y) \simeq H_1(X \cap Y).$$
Exercise 2.3. Write down all the differentials above carefully, as well as the induced maps between the rows, checking that the diagram commutes. Then calculate the homology directly.

2.2. Relative Homology. When $X \subseteq Y$, a basic topological question is how the homology of $X$ and $Y$ are related. At the level of simplicial complexes, $C_i(X) \subseteq C_i(Y)$ so the most straightforward approach is to consider the short exact sequence of complexes whose $i^{th}$ term is

$$0 \rightarrow C_i(X) \rightarrow C_i(Y) \rightarrow C_i(Y)/C_i(X) \rightarrow 0$$

Definition 2.4. The $i^{th}$ relative homology $H_i(Y,X)$ of the pair $X \subseteq Y$ is the homology of the quotient complex whose $i^{th}$ term is $C_i(Y)/C_i(X)$.

By Theorem 1.6 there is a long exact sequence in relative homology

$$\cdots \rightarrow H_{n+1}(Y,X) \rightarrow H_n(X) \rightarrow H_n(Y) \rightarrow H_n(Y,X) \rightarrow H_{n-1}(X) \rightarrow \cdots$$

Example 2.5. We use the complex of Example 2.2, using as our “big” space $X \cup Y$ and as our “small” space $X \cap Y$, yielding a short exact sequence of complexes

\[
\begin{array}{ccccccccc}
C_\bullet(X \cap Y) : & 0 & \rightarrow & 0 & \rightarrow & \mathbb{R}^3 & \rightarrow & R^3 & \rightarrow & 0 \\
C_\bullet(X \cup Y) : & 0 & \rightarrow & \mathbb{R}^6 & \rightarrow & \mathbb{R}^9 & \rightarrow & \mathbb{R}^9 & \rightarrow & \mathbb{R}^5 & \rightarrow & 0 \\
C_\bullet(X \cup Y)/C_\bullet(X \cap Y) : & 0 & \rightarrow & \mathbb{R}^6 & \rightarrow & \mathbb{R}^6 & \rightarrow & \mathbb{R}^6 & \rightarrow & \mathbb{R}^2 & \rightarrow & 0 \\
\end{array}
\]

The horizontal differentials in the top two rows are the same as the differentials $\partial_i^\cap$ and $\partial_i^\cup$ in Example 2.2, and the vertical map from the first row to the second row is the inclusion map. Example 2.2 showed that $H_2(X \cup Y) \simeq \mathbb{R}^1$ and in Exercise 2.3, we calculated that $H_1(X \cup Y) = 0$. The long exact sequence yields

$$0 \rightarrow H_2(X \cup Y) \rightarrow H_2(Y,X) \rightarrow H_1(X \cap Y) \rightarrow H_1(X \cup Y) = 0.$$ 

Hence $H_2(Y,X) \simeq \mathbb{R}^2$. In relative homology, the “belt” $X \cap Y \simeq S^1$ around the “belly” of $X \cup Y \simeq S^2$ is tightened down to a point, resulting in a pair of $S^2$’s.

Exercise 2.6. Write out the differentials $\partial_i^{rel}$ and the vertical maps, check everything commutes, and compute homology. 


3. Cohomology: Simplicial, Čech, de Rham theories

For a complex

$$\mathcal{C} : 0 \rightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots$$

we have $\text{im}(d^i) \subseteq \ker(d^{i+1})$, and the cohomology of $\mathcal{C}$ is defined as

$$H^i(\mathcal{C}) = H^i \left( 0 \rightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \right)$$

So cohomology is homology with increasing indices. However, in a number of common situations, there is a multiplicative structure on the set of $C^i$. This results in a multiplicative structure on cohomology, yielding a ring $H^* (\mathcal{C}) = \oplus_i H^i (\mathcal{C})$.

3.1. Simplicial Cohomology. For a simplicial complex $\Delta$ and coefficient ring $R$, we have the complex of $R$-modules

$$\mathcal{C}_* = \cdots \rightarrow C_{i+1}(\Delta, R) \xrightarrow{d_{i+1}} C_i(\Delta, R) \xrightarrow{d_i} C_{i-1}(\Delta, R) \rightarrow \cdots$$

Applying $\text{Hom}_R(\bullet, R)$ and defining $C^i(\Delta, R) = \text{Hom}_R(C_i(\Delta), R)$ yields

$$\mathcal{C}^* = \cdots \leftarrow C^{i+1}(\Delta, R) \xleftarrow{d_i} C^i(\Delta, R) \xleftarrow{d_{i-1}} C^{i-1}(\Delta, R) \leftarrow \cdots$$

Elements of $C_i$ are called $i$-chains, and elements of $C^i$ are called $i$-cochains.

**Definition 3.1.** The multiplicative structure (called cup product) on simplicial cohomology is induced by a map on the cochains:

$$C^i(\Delta) \times C^j(\Delta) \xrightarrow{\cup} C^{i+j}(\Delta).$$

To define such a map we must define an action of a pair $(c_i, c_j) \in C^i(\Delta) \times C^j(\Delta)$ on an element of $C_{i+j}(\Delta)$: if $[v_0, \ldots, v_{i+j}] = \sigma \in C_{i+j}(\Delta)$, then

$$(c_i, c_j)[v_0, \ldots, v_{i+j}] = c_i[v_0, \ldots, v_i] \cdot c_j[v_i, \ldots, v_{i+j}],$$

where the last $\cdot$ is multiplication in the ring $R$.

**Exercise 3.2.** Compute the simplicial cohomology rings for the complexes $X \cap Y$ and $X \cup Y$ of Example 2.2, with coefficients in a field $\mathbb{K}$. From a topological standpoint,

$$X \cap Y \sim S^1 \text{ and } X \cup Y \sim S^2.$$ 

You should find $H^1(X \cup Y) = 0$ and

$$H^0(X \cap Y) \cong \mathbb{K} \cong H^1(X \cap Y) \text{ and } H^0(X \cup Y) \cong \mathbb{K} \cong H^2(X \cup Y)$$

The interesting part of the exercise is determining the multiplicative structure. ◊

**Exercise 3.3.** For the ambitious: consult a book on algebraic topology and find a triangulation of the torus $T^2$ (the surface of a donut). Show the cohomology ring with $\mathbb{K}$ coefficients is $\mathbb{K}[x, y] / \langle x^2, y^2 \rangle$. Now do the same thing for the real projective plane $\mathbb{R}P^2$, first with $\mathbb{K} = \mathbb{Q}$ and then with $\mathbb{K} = \mathbb{Z}/2$. You should get different answers; in particular the choice of field matters. This stems from the fact that the real projective plane is not orientable. ◊
3. Čech Cohomology. For a sheaf $\mathcal{F}$, Čech cohomology is constructed so that the global sections of $\mathcal{F}$ defined in §3 of Chapter 4 are given by $\check{H}^0(\mathcal{F})$. Let $\mathcal{U} = \{U_i\}$ be an open cover of $X$. If $\mathcal{U}$ consists of a finite number of open sets, then the $i^\text{th}$ module $C^i$ in the Čech complex is simply

$$C^i(\mathcal{F}, \mathcal{U}) = \bigoplus_{\{j_0 < \ldots < j_i\}} \mathcal{F}(U_{j_0} \cap \ldots \cap U_{j_i}).$$

In general a cover need not be finite; in this case it is convenient to think of an element of $C^i$ as an operator $c_i$ which assigns to each $(i + 1)$-tuple $(j_0, \ldots, j_i)$ an element of $\mathcal{F}(U_{j_0} \cap \ldots \cap U_{j_i})$. We build a complex $C^\bullet$ via:

$$C^i = \prod_{\{j_0 < \ldots < j_i\}} \mathcal{F}(U_{j_0} \cap \ldots \cap U_{j_i}) \xrightarrow{d^i} C^{i+1} = \prod_{\{j_0 < \ldots < j_{i+1}\}} \mathcal{F}(U_{j_0} \cap \ldots \cap U_{j_{i+1}}),$$

where $d^i(c_i)$ is defined by how it operates on $(i + 2)$-tuples, which is:

$$d^i(c_i)(j_0, \ldots, j_{i+1}) = \sum_{k=0}^{i+1} (-1)^k c_i(j_0, \ldots, \hat{j}_k, \ldots, j_{i+1})|_{\mathcal{F}(U_{j_0} \cap \ldots \cap U_{j_{i+1}})}.$$

**Definition 3.4.** The $i^\text{th}$ Čech cohomology of $\mathcal{F}$ and $\mathcal{U}$ is $H^i(\mathcal{U}, \mathcal{F}) = \check{H}^i(\mathcal{C}^\bullet)$.

**Exercise 3.5.** The constant sheaf on a space $X$ is defined by giving $\mathbb{Z}$ the discrete topology, and defining $\mathbb{Z}(U)$ as the space of continuous functions from $U$ to $\mathbb{Z}$.

(a) Is it true that $\mathbb{Z}(U) \simeq \mathbb{Z}$ for any open set $U$? If not, what additional assumption on $U$ would make this true?

(b) Compute the Čech cohomology of the constant sheaf $\mathbb{Z}$ on $S^2$ using the cover of the open top hemisphere, and the two open bottom “quarterspheres” (all opens slightly enlarged so that they overlap) and see what you get. Your chain complex should start with these three opens (each of which is topologically an $\mathbb{R}^2$), then the three intersections (each of which is again an $\mathbb{R}^2$) and then the triple intersection, which is two disjoint $\mathbb{R}^2$'s.

(c) For the simplicial complex of Example 2.2 write down the Čech complex corresponding to open sets which are the triangular faces, slightly enlarged so they overlap. Compare your result to Example 2.2.

(d) Find the Čech complex for the open cover of $S^2$ corresponding to the top hemisphere and bottom hemisphere, and compute the Čech cohomology. Why is your answer different from part (c)?

(e) Let $X$ be a topological space with finite triangulation $\Delta$, and create an open cover $\mathcal{U}$ by taking $\epsilon$-enlarged neighborhoods of each face of $\Delta$. Suppose that for every intersection $U = \cap U_i$ of opens, $U$ is contractible. Then $\check{H}^i(U, \mathbb{Z}) = 0$ for all $i \geq 1$, and Čech cohomology with $\mathbb{Z}$-coefficients agrees with the topological cohomology $H^i(\Delta, \mathbb{Z})$.

A cover as in part (e) is known as a *Leray cover*. See Theorem 4.5 in Chapter 9.
The definition of Čech cohomology is concrete and computable for a given cover, but part (d) of Exercise 3.5 illustrates a fatal flaw: different covers yield different results. This can be fixed by the following construction:

**Definition 3.6.** Order the set of open covers of $X$ via $\mathcal{V} \preceq \mathcal{U}$ if $\mathcal{U}$ is finer than $\mathcal{V}$: for any $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ with $U \subseteq V$. This gives the set of all covers the structure of a poset, hence we can take the direct limit. The $i^{th}$ Čech cohomology of $\mathcal{F}$ is

$$\check{H}^i(\mathcal{F}) = \lim_{\longrightarrow} H^i(\mathcal{U}, \mathcal{F})$$

While this is not a useful definition from a computational standpoint, the criterion of Leray appearing above gives sufficient conditions for a cover to compute $\check{H}^i(\mathcal{F})$. If all intersections of the open sets of a cover have no cohomology except at the zeroth position, then

$$\check{H}^i(\mathcal{F}) = \check{H}^i(\mathcal{U}, \mathcal{F})$$

For sheaves of modules $\mathcal{F}_i$ on a smooth variety $X$, we have the following key theorem:

**Theorem 3.7.** Given a short exact sequence of sheaves

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0,$$

there is a long exact sequence in sheaf cohomology

$$\cdots \rightarrow H^{i-1}(\mathcal{F}_3) \rightarrow H^i(\mathcal{F}_1) \rightarrow H^i(\mathcal{F}_2) \rightarrow H^i(\mathcal{F}_3) \rightarrow H^{i+1}(\mathcal{F}_1) \rightarrow \cdots .$$

The proof rests on an alternate approach to defining sheaf cohomology, treated in Chapter 9. Here is a quick sketch: the global section functor is left exact and covariant. Using the derived functor approach of Chapter 9, we can take an injective resolution of $\mathcal{O}_X$-modules. By the derived functor machinery, a short exact sequence of sheaves yields a long exact sequence in the higher derived functors of $H^0$. Showing that these higher derived functors agree with the Čech cohomology would suffice to prove the theorem. This (along with the existence of an injective resolution) requires some work, and can be found in Section III.4 of [62]. The most important instance of a short exact sequence of sheaves is the ideal sheaf sequence.

For a variety $X$ sitting inside $\mathbb{P}^n$, say $X = V(I)$, we have the usual exact sequence of modules:

$$0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0.$$

Since exactness is measured on stalks, if we take an exact sequence of modules and look at the associated sheaves, we always get an exact sequence of sheaves. So there is an exact sequence of sheaves on $\mathbb{P}^n$:

$$0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_X \rightarrow 0.$$
3.3. de Rham Cohomology. We saw in Chapter 4 that sheaves are built to encode local information, along with gluing data that allows us to translate on overlaps between local charts. A quintessential example is the sheaf of differential $p$-forms on a smooth manifold $X$: for a sufficiently small open set $U \simeq \mathbb{R}^n$ and local coordinates $\{x_1, \ldots, x_n\}$ on $U$,

$$\Omega^p_X(U) = \{ f(x_1, \ldots, x_n) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p} \},$$

where $f(x_1, \ldots, x_n)$ is a $C^\infty$ function on $U$. For $p = 0$ we have

$$\Omega^0_X(U) = C^\infty(U).$$

These are familiar objects from vector calculus: on a surface $S$ we integrate

$$f(s, t)dsdt,$$

which is an element of $\Omega^2_S$.

The geometry of integration serves as a guide to what is going on here. Differentials measure volume: a two-form $ds \cdot dt$ represents an infinitesimal rectangle with sides $ds$ and $dt$. The area of a parallelogram in $\mathbb{R}^2$ with sides given by vectors $v$ and $w$ is measured by the determinant, which is essentially what the two form represents. Orientation matters:

$$\det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} = - \det \begin{bmatrix} w_1 & v_1 \\ w_2 & v_2 \end{bmatrix}$$

When we take orientation into account (as opposed to just the area $ds \cdot dt$), we write $ds \wedge dt$, with the relation

$$ds \wedge dt = -dt \wedge ds,$$

coming from the determinant. This is the key to the main theorem of vector calculus: Stokes’ theorem generalizes to higher dimensions, and the intuition remains the same: if $\omega \in \Omega^p_S$ and $S$ is a closed, bounded $p+1$ dimensional region, then the boundary $\partial(S)$ is $p$-dimensional, and under suitable conditions

$$\int_S d\omega = \int_{\partial(S)} \omega.$$

Exercise 3.8. For $X = \mathbb{R}^3$, differentiation takes a function $f(x_1, x_2, x_3)$ to the gradient $\nabla f$, which is a one-form:

$$\Omega^0_X \xrightarrow{d} \Omega^1_X \text{ via } f \mapsto \sum_{i=1}^3 \frac{\partial f}{\partial x_i} dx_i.$$

There are similar interpretations for $d\omega$ when $\omega \in \Omega^1_X$ or $\Omega^2_X$ in terms of curl and divergence. Try working them out; if you get stuck, Spivak [93] will help. ◊
Differentiation takes $p$-forms to $p+1$-forms, hence there is a sequence
\[ \cdots \rightarrow \Omega^p_X \xrightarrow{d} \Omega^{p+1}_X \rightarrow \cdots \]
Let
\[ I_p = \{ 1 \leq j_1 < j_2 < \cdots < j_p \leq n \}, \]
and for $P \in I_p$ let $dP$ denote the $p$-form $dx_{j_1} \wedge \cdots \wedge dx_{j_p}$. The key is that differentiating a $p$-form twice yields zero:
\[ d^2 \left( \sum_{P \in I_p} f_PD_P \right) = 0. \]

**Exercise 3.9.** Prove this as follows: fix a $\omega = f \cdot dx_{j_1} \wedge \cdots \wedge dx_{j_p}$, and examine the coefficients which appear in $d^2(\omega)$. This is the higher dimensional version of a result from vector calculus:
\[ \text{div(curl}(F)) = 0 \]
The proof should remind you (exactly the same bookkeeping is involved) of the proof that the simplicial boundary operator $\partial$ satisfies $\partial^2 = 0$. To emphasize the fact that the order of indexing the $p$-forms matters, the operation of differentiation of $p$-forms is called *exterior differentiation*. $\diamond$

The *exterior algebra* provides a formal framework that is the right setting in which to investigate exterior differentiation and differential forms:

**Definition 3.10.** Let $V \simeq \mathbb{K}^n$ be a finite dimensional vector space. In Definition 2.14 of Chapter 2 the exterior algebra $\Lambda(V)$ was defined as a quotient of the tensor algebra $T(V)$ by the relations
\[ \{ v_i \otimes v_j + v_j \otimes v_i \}. \]
A quotient space has many guises; one way to obtain a concrete realization is to choose a basis $\{ e_1, \ldots, e_n \}$ for $V$. Then $\Lambda(V)$ is a $2^n$ dimensional graded vector space. The relation $e_i \otimes e_j + e_j \otimes e_i = 0$ propagates to yield
\[ e_{i_1} \wedge \cdots \wedge e_{i_p} = (-1)^{\text{sgn}(\sigma)} e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_p)} \text{ for } \sigma \in S_n. \]
Thus, a basis for the $p^{th}$ graded component $\Lambda^p(V)$ of $\Lambda(V)$ is given by
\[ \{ e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leq i_1 < i_2 < \cdots < i_p \leq n \}. \]
The exterior algebra on $V \simeq \mathbb{K}^n$ behaves *almost* exactly like the polynomial ring in $n$ variables, but where $x_ix_j = -x_jx_i$. Hence, $x_i^2 = 0$ for all $i \in \{ 1, \ldots, n \}$ and all monomials in $\Lambda(V)$ are squarefree.

**Example 3.11.** For $V = \mathbb{K}^3$, a basis for $\Lambda(V)$ is
\[ \{ 1, e_1, e_2, e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_3 \} \]
Multiplication in $\Lambda(V)$ is called wedge product and written as $\wedge$. To determine the appropriate sign we must count the number of transpositions. So for example
\[ e_2 \wedge (e_1 \wedge e_3) = -e_1 \wedge e_2 \wedge e_3 \text{ whereas } e_3 \wedge (e_1 \wedge e_2) = e_1 \wedge e_2 \wedge e_3. \]

**Exercise 3.12.** Write out the multiplication table for $\Lambda(\mathbb{K}^3)$, ignoring the identity. What can you say about the relation between the symmetric and exterior algebras when the field $\mathbb{K}$ has characteristic two? ⋄

There is a point to make here: the exterior algebra of a vector space $V \simeq \mathbb{K}^n$ is constructed in a purely formal fashion. If $R = \mathbb{K}[x_1, \ldots, x_m]$ and we replace $V$ with the free module $R^n$, the same increase in complexity that occurred in passing from vector spaces to modules manifests itself. In particular, $\Lambda(R^n)$ comes with the additional operation of exterior differentiation, whereas $\Lambda(\mathbb{K}^n)$ has no such operation. It is by passing to a more complicated algebraic object that we make it possible for geometry to enter the picture.

**Definition 3.13.** Exercise 3.9 shows that applying the exterior differentiation operator twice yields zero, and Georges de Rham used this insight to define an eponymous cohomology theory:

\[ H^p_{dR}(X) = H^p\left( \cdots \rightarrow \Omega^p_X \xrightarrow{d} \Omega^{p+1}_X \rightarrow \cdots \right) \]

We seem to have strayed very far afield from the comfortable combinatorial setting of simplicial complexes and simplicial homology. But that is not the case at all!

**Theorem 3.14.** [de Rham Theorem] Let $X$ be a smooth manifold, and $\Delta$ a triangulation of $X$. Then the simplicial and de Rham cohomology groups are isomorphic:

\[ H^p_{dR}(X) \simeq H^p(\Delta). \]

**Proof.** As in the proof of the snake lemma, the key is to define the right map, which we do at the level of cochains. A map
\[ C^p_{dR} \rightarrow C^p(\Delta), \]
takes as input a $p$-form $\omega$, and produces an element of $\text{Hom}_R(C_p(\Delta), \mathbb{R})$. Let $\sigma \in C_p(\Delta)$. Since integration is a linear operator, we have a pairing
\[ \langle \omega, \sigma \rangle = \int_\sigma \omega. \]
It is not an isomorphism at the level of chains, but it is at the level of cohomology; see [90] for a detailed proof. □
4. Ranking and Hodge Theory

Hodge theory is a central area of geometry, focused on the study of differential forms; see Voisin [95] for a comprehensive treatment. This section describes an application of Hodge theory to data science. The Netflix problem asks for a global ranking of objects, when individual input ranking data is inconsistent. In particular, the rankings provided as input may contain cycles $a > b > c > a$. We start off by giving an overview of Hodge theory, then derive a structure theorem (the Hodge decomposition) in the simplest possible setting. We close with a sketch of the approach of [66] to the Netflix problem via the Hodge decomposition.

4.1. Hodge Decomposition. In the setting of the previous section, we have

**Theorem 4.1.** [Hodge Theorem] For a smooth compact manifold $X$, there is a decomposition

$$
\Omega^k_X \cong \text{im}(d^{k-1}) \oplus \text{im}(d^k) \oplus \ker(L)
$$

where $L$ is the Laplacian, and

$$
H^k_{dR}(X) \cong \ker(L).
$$

Before we unpack the terms in the formula, we highlight the main consequence: homology and cohomology are represented as quotient spaces, so there is almost never any hope of finding canonical representatives. The Hodge theorem says this is not the case for $H^k_{dR}(X)$ when $X$ is smooth and compact: the kernel of the Laplacian provides exactly such canonical representatives. The operator $d^{k-1}$ in the formula above is exterior differentiation, the operator $d^k$ is the adjoint operator

$$
\Omega^{k+1}_X \xrightarrow{d^k} \Omega^k_X,
$$

and the Laplacian $L$ is defined as the operator

$$
d^k \circ d^k + d^{k-1} \circ d^{k-1}.
$$

The first term $d^k \circ d^k$ of the Laplacian “goes forward then comes back”, while the second term $d^{k-1} \circ d^{k-1}$ “goes backwards then goes forward”. An important ingredient in the proof (which uses elliptic PDE, and is beyond the scope of these notes) is the Hodge star operator. In the case of the exterior algebra on $V \simeq \mathbb{K}^n$, we have

$$
\Lambda^i(V) \simeq \oplus \Lambda^j(V)
$$

and Hodge star identifies $V^k \perp$ with $V^{n-k}$ via the pairing

$$
V^k \times V^{n-k} \to V^n \simeq \mathbb{K} \text{ via } (v, w) \mapsto v \wedge w.
$$

Stripping down to the bare essentials of the vector space setting, we next give a lowbrow proof of the Hodge decomposition.
4. Ranking and Hodge Theory

**Definition 4.2.** Let $V, W$ be finite dimensional inner product spaces, and

$$ V \xrightarrow{A} W $$

a linear transformation. For concreteness, let $\mathbb{K} = \mathbb{R}$, with inner product given by the usual dot product, and choose bases so that $A$ is a matrix. The adjoint operator $A^*$ is defined via

$$ \langle Av, w \rangle = \langle v, A^*w \rangle. $$

So in this setting, $A^T = A^*$, and we have

$$ \langle Av, w \rangle = w^T \cdot Av = (Av)^T \cdot w = v^T \cdot A^T w = \langle v, A^*w \rangle. $$

**Proposition 4.3.** Let

$$ V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} V_3 $$

be a complex of vector spaces, with $\text{rank}(V_i) = a_i$. Then

$$ V_2 \cong \text{im}(d_1) \oplus \text{im}(d_2^T) \oplus \ker(L), \text{ where } L = d_1d_1^T + d_2^T d_2. $$

**Proof.** Let $\text{rank}(d_i) = r_i$ and $\dim \ker(d_i) = k_i$. Choose bases so that

$$ d_1 = \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad d_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{r_2} \end{bmatrix} $$

where 0 represents a zero matrix of the appropriate size. For example, since the matrix $d_1$ is $a_2 \times a_1 = a_2 \times (r_1 + k_1)$, the top right zero in $d_1$ has $r_1$ rows and $k_1$ columns. Then we have that $d_1d_1^T$ and $d_2^T d_2$ are both $a_2 \times a_2$ matrices, with

$$ d_1d_1^T = \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad d_2^T d_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{r_2} \end{bmatrix} $$

Hence

$$ L = d_1d_1^T + d_2^T d_2 = \begin{bmatrix} I_{r_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{r_2} \end{bmatrix} $$

The zero in position $(2, 2)$ represents a square matrix of size

$$ k_2 - r_1 = \dim(\ker(d_2)/\text{im}(d_1)),$$

and the linear algebra version of the Hodge decomposition follows. \qed
4.2. Application to Ranking. In [66], Jiang-Lim-Yao-Ye apply the Hodge decomposition to the Netflix problem. Viewers rank movies; producing a global ranking is complicated by the presence of loops. We begin by creating a weighted directed graph $G$, where a weight of $m$ on the directed edge $[a, b]$ represents an aggregate of $m$ voters preferring movie $[a]$ to movie $[b]$. Construct a two dimensional simplicial complex $\Delta_G$ by adding in the face $[abc]$ if all edges of the triangle are present. This is an instance of a flag complex: for any $K_m$ subgraph in $\Delta_1$, the flag complex contains the corresponding $m - 1$ simplex.

The assignment of weights to each edge defines a linear functional on $C_1(\Delta_G)$, which by Proposition 4.3 has a decomposition

$$C_1(\Delta_G, \mathbb{R}) \cong \text{im}(d_T^1) \oplus \text{im}(d_2) \oplus \ker(L)$$

Cycles of any length represent inconsistencies, so the global ranking should ignore them. Cycles of length three have been filled in to create two-simplices, which are represented by $\text{im}(d_2)$. Cycles of length four or more correspond to homology classes, so are represented by $\ker(L)$. Hence we have

**Theorem 4.4.** [Jiang-Lim-Yao-Ye] For inconsistent ranking data as in the Netflix problem, orthogonal projection onto the subspace $\text{im}(d_T^1)$ produces the global ranking which most closely reflects the voter preferences.

**Proof.** The Hodge decomposition. □

Since we introduced flag complexes above, it seems worth mentioning the most famous open question on flag complexes: the Charney-Davis [26] conjecture.

**Conjecture 4.5.** Let $\Delta$ be a flag complex which triangulates an odd dimensional sphere $S^{2d-1}$, and let $f_{i-1}$ denote the number of $i - 1$ dimensional faces of $\Delta$. Then

$$(-1)^d \sum_{i=0}^{2d} \left( \frac{-1}{2} \right)^i f_{i-1} \geq 0.$$ 

Davis-Okun prove the $d = 2$ case in [35] using $\ell^2$-cohomology.
Persistent Homology

We’re ready to reap the fruits of our labors in the previous chapters, and attack the motivating problem from the preface: given point cloud data $X \subseteq \mathbb{R}^n$, we want to extract meaning from $X$. In this chapter, we study persistent homology (PH), which can facilitate detection and analysis of underlying structure in large datasets. Persistent homology assigns a module over a principal ideal domain to a filtered simplicial complex. In Chapter 3, we proved that a finitely generated module $M$ over a principal ideal domain $R$ has a decomposition:

$$M \cong R^m \bigoplus_{i=1}^{k} R/p_i.$$ 

Persistent homology is often displayed as a barcode diagram. The decomposition above translates into a barcode as follows: the free summands (which are known as long bars) represent persistent features of the underlying data set, while the torsion components (which are known as short bars) represent noise in the data. Persistent homology has proven effective in a number of settings, including breast cancer pathology [19], viral evolution [25], and visual cortex activity [91], to name just a few. We start by using $X$ as a seed from which to build a family of spaces

$$X_\epsilon = \bigcup_{p \in X} N_\epsilon(p),$$

where $N_\epsilon(p)$ denotes an $\epsilon$ ball around $p$.

As $X_\epsilon \subseteq X_{\epsilon'}$ if $\epsilon \leq \epsilon'$, the result is a filtered topological space. This chapter covers

- Čech Complex and Rips Complex.
- Morse Theory.
- Categories and Functors.
- Persistent Homology.
1. Čech Complex and Rips Complex

The first step in constructing persistent homology is to pass from the filtered topological space to a filtered simplicial complex. So for each snapshot $X_\epsilon$, we would like a corresponding $\Delta_\epsilon$. We begin with a standard topological construction

**Definition 1.1.** For an open cover $\mathcal{U}$ of a space $X$ with index set $I$, let $I_k$ denote the set of all $k+1$-tuples of $I$. The nerve $N(\mathcal{U})$ of $\mathcal{U}$ is a simplicial complex whose $k$-faces $N(\mathcal{U})_k$ are given by

$$N(\mathcal{U})_k \iff \{ \alpha_k \in I_k \mid (\bigcap_{i \in \alpha_k} U_i) \neq \emptyset \}$$

To see that $N(\mathcal{U})$ is a simplicial complex, note that since if $\alpha_k \in I_k$ and $\beta_{k-1}$ is a $k$-subset of $\alpha_k$, this implies that

$$\left( \bigcap_{i \in \beta_{k-1}} U_i \right) \neq \emptyset$$

The nerve is important due to the

**Theorem 1.2.** [Nerve Theorem] If $\mathcal{U}$ is a finite open cover of $X$ such that all intersections $U$ of the open sets of $\mathcal{U}$ are contractible, then $N(\mathcal{U})$ is homotopic to $X$.

Therefore in the setting of the Nerve Theorem,

$$H^i(N(\mathcal{U}), \mathbb{Z}) \simeq H^i(X, \mathbb{Z}).$$

This can be proved using Theorem 4.5 from Chapter 9; see also Exercise 3.5 of Chapter 6. In the setting of persistent homology, there are two standard ways to build $\Delta_\epsilon$; the first method mirrors the construction of the nerve of a cover.

**Definition 1.3.** The Čech complex $\mathcal{C}_\epsilon$ has $k$-simplices which correspond to $k+1$-tuples of points $p \in X$ such that

$$\bigcap_{i=0}^{k} N_{\epsilon}(p_i) \neq 0$$

The $k+1$ closed balls of radius $\frac{\epsilon}{2}$ centered at $\{p_0, \ldots, p_k\}$ share a common point.

It follows from Theorem 1.2 that

**Theorem 1.4.** There is a homotopy $\mathcal{C}_\epsilon \simeq X_{\frac{\epsilon}{2}}$.

**Definition 1.5.** The Rips (or Rips-Vietoris) complex $\mathcal{R}_\epsilon$ has $k$-simplices which correspond to $k+1$-tuples of points $p \in X$ such that

$$d(p_i, p_j) \leq \epsilon$$

for all pairs $i, j$. 

The Rips complex is often more manageable from a computational standpoint, while the Čech complex often makes proofs simpler. In [37], de Silva-Ghrist show that there is a nice relationship between the two:

**Theorem 1.6.** There are inclusions

\[ \mathcal{R}_\epsilon \hookrightarrow \mathcal{C}_{\sqrt{2}\epsilon} \hookrightarrow \mathcal{R}_{\sqrt{2}\epsilon} \]

**Example 1.7.** For the four points \{ \((0, 0), (1, 0), (0, 1), (1, 1)\) \} we compute the Čech and Rips complexes. For \(\epsilon < 1\) both \(\mathcal{C}_\epsilon\) and \(\mathcal{R}_\epsilon\) consist only of the points themselves. For \(\epsilon = 1\), we have

\[ \bigcup_{i=1}^{4} N_{1/2}(p_i) \text{ is } \]

Hence \(\mathcal{C}_1\) and \(\mathcal{R}_1\) are both (hollow) squares

and both complexes remain hollow squares until \(\epsilon = \sqrt{2}\), at which time both become (solid) tetrahedra.

**Exercise 1.8.** Let \(X_n\) consist of vertices \(\{p_1, \ldots, p_n\}\) of a symmetric \(n-gon\), with \(d(0, p_i) = 1\). Carry out the computation of Example 1.7 for \(X_n\).
2. Morse Theory

Persistent homology is similar to Morse theory, and to understand persistent homology it will be useful to have Morse theory as a backdrop. Morse developed the machinery almost a century ago in [78], [79]. We give a brief overview here in the spirit of [13] or [98]; for a more comprehensive treatment, see [73] or [76].

Definition 2.1. A smooth function $M \xrightarrow{f} \mathbb{R}$ from an $n$-dimensional manifold $M$ to $\mathbb{R}$ is defined on open sets $U \subseteq M$ having local coordinates $\{x_1, \ldots, x_n\}$. The function $f$ is \textit{Morse} if it has a finite number of critical points
\[
\frac{\partial f}{\partial x_1}(p) = \cdots = \frac{\partial f}{\partial x_n}(p) = 0,
\]
and the critical points are distinct and \textit{nondenerate}: the determinant of the Hessian matrix
\[
H_f = \begin{bmatrix}
\frac{\partial^2 f}{\partial x_1^2}(p) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(p) \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1}(p) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(p)
\end{bmatrix}
\]
is nonzero at critical points. Morse theory generalizes the second derivative test.

Example 2.2. Flashback to vector calculus: the graph of $z = f(x, y)$ defines a surface $S$ in $\mathbb{R}^3$. The critical points of $f$ are the points $p$ where
\[
\frac{\partial f}{\partial x}(p) = 0 = \frac{\partial f}{\partial y}(p),
\]
so the tangent plane $T_p(S)$ is horizontal and $p$ is a max, min, or saddle point.

Exercise 2.3. Find the critical points and Hessians for the following functions
(a) $z = x^2 + y^2$.
(b) $z = x^2 - y^2$.
(c) $z = -x^2 - y^2$.

Determine if the critical point is a max, min, or saddle.

A key ingredient in Morse Theory is the Morse Lemma:

Lemma 2.4. For a Morse function $f$, if $p$ is a critical point, then there exists a coordinate system $\{x_1, \ldots, x_n\}$ so that
\[
f(x_1, \ldots, x_n) = f(p) - \sum_{i=1}^{\lambda_p} x_i^2 + \sum_{j=\lambda_p+1}^{n} x_j^2.
\]
It follows immediately from the Morse Lemma that at a critical point $p$, the Hessian matrix is diagonal, with $\lambda_p$ entries of $-2$, and $n - \lambda_p$ entries of $+2$. So $\lambda_p$ is the number of negative eigenvalues of the Hessian at a critical point $p$. 

\diamond
Define sublevel sets of the manifold \( M \) via
\[
M_r = f^{-1}(-\infty, r]
\]
The topology of the sublevel sets \( M_r \) only changes at a finite number of points.

This agrees with our vector calculus intuition: if the tangent plane \( T_p(S) \) to a smooth, nonplanar surface \( S \) is horizontal at \( p \) with equation \( z = c \), then the plane \( z = c + \epsilon \) either (locally near \( p \)) misses \( S \), or intersects \( S \) in a curve. This dichotomy—that behavior changes at critical points—is the content of the famous Morse Theorem:

**Theorem 2.5.** Let \( r < r' \in \mathbb{R} \) and \( f \) be a Morse function on \( M \). Then

(a) If there are no critical points in \([r, r']\), then \( M_r \) and \( M_{r'} \) are homotopic.

(b) If \( r' \) is the least value greater than \( r \) such that \( p = f^{-1}(r') \) is a critical point, then
\[
M_{r'} \simeq M_r \cup B_{\lambda_p},
\]
with \( B_{\lambda_p} \) a ball of dimension \( \lambda_p \), along with an attaching map sending
\[
\partial(B_{\lambda_p}) \to M_r.
\]

A few remarks are in order. First, the homotopy in item (a) is a deformation retract: there is a continuous map \( M_{r'} \to M_r \) leaving \( M_r \) fixed.

**Example 2.6.** A two-sphere punctured at the north pole deformation retracts to a point—the south pole—by sliding every point south along latitude lines.

If \( r < r' \) with \( r' = f(p) \) the smallest value of a critical point greater than \( r \), then for \( 0 < \epsilon \leq r' - r \), \( M_r \) is homotopic to \( M_{r' - \epsilon} \). The attaching map can be thought of as gluing \( B_{\lambda_p} \) in to \( M_{r' - \epsilon} \) at the moment \( \epsilon \) becomes zero.
Theorem 2.7. Let \( f \) be a Morse function on \( M \), and \( p = f^{-1}(r') \) a critical point. Suppose \( r < r' \) and there are no critical points in \( f^{-1}([r, r')) \). Then either

(a) \( \dim H_{\lambda_p-1}(M_r) = \dim H_{\lambda_p-1}(M_r) - 1 \), or
(b) \( \dim H_{\lambda_p}(M_{r'}) = \dim H_{\lambda_p}(M_r) + 1 \).

Both possibilities are consequences of item (b) in Theorem 2.5. The intuition is that gluing in a ball \( B_{\lambda_p} \) can either fill in a \( \lambda_p - 1 \) dimensional hole, or be the final brick sealing off a hollow chamber of dimension \( \lambda_p \) (hopefully without Fortunato–in pace requiescat–entombed within).

Example 2.8. A two-dimensional ball is a disk, and gluing in a disk could

- Fill in a circle, eliminating an \( S^1 \), as in case (a):

![Diagram of filling in a circle](image1)

\( H_1 \) drops in dimension.

- Put a lid on an open bowl, creating an \( S^2 \), as in case (b):

![Diagram of creating a lid](image2)

\( H_2 \) increases in dimension.

Corollary 2.9. For a smooth manifold \( M \) and Morse function \( f \), let \( c_i \) be the number of critical points \( p \) of \( f \) such that \( \lambda_p = i \). Then

\[ c_i \geq \dim H_i(M). \]

Proof. The number \( c_i \) bounds the number of \( i \)-dimensional cells in the CW complex used to construct \( M \), hence also the \( i \)th homology. \( \square \)

Example 2.10. For the torus \( T^2 \), at \( r < \frac{1}{4} \), \( f \) is given locally by a function close to \( x^2 + y^2 \), and so \( \lambda_p = 0 \), similarly for \( r > \frac{3}{4} \) the function \( f \) is given locally by a function close to \( -x^2 - y^2 \) so \( \lambda_p = 2 \). At the two saddle points \( H_f \) has one negative eigenvalue, so the vector of \( c_i \) is \((1, 2, 1)\).
3. Categories and Functors

Category Theory is an abstract framework that encompasses homology, persistent homology, and constructions such as $\text{Hom}$, $\otimes$, and localization. Carried to extremes, category theory can become “the most arid of intellectual pursuits” \[85\]. Used judiciously, it provides a key tool to streamline and unify many mathematical constructions. In the next section we’ll see that persistent homology can be regarded as a functor from the Poset category to a category $\mathcal{C}$. Below we give a brief introduction to categories and functors; the material is developed in more depth when we introduce derived functors in Chapter 9.

**Definition 3.1.** A Category $\mathcal{A}$ consists of
- A collection of objects $\text{Ob}(\mathcal{A})$.
- A collection of morphisms: for $A, B \in \text{Ob}(\mathcal{A})$, a set $\text{Hom}(A, B)$, such that morphisms can be composed. The composition is associative, and for each object $A$, there is an identity morphism $1_A : A \to A$.

**Example 3.2.** Examples of categories
- **Set**: Sets and functions.
- **P**: Posets and order preserving functions.
- **Vect**: Vector spaces and linear transformations.
- **Top**: Topological spaces and continuous maps.

In many mathematical settings we deal with objects and maps between them. A map between categories is

**Definition 3.3.** A functor $F$ is a function from a category $\mathcal{A}$ to a category $\mathcal{B}$, which takes objects to objects

$$\text{Ob}(\mathcal{A}) \xrightarrow{F} \text{Ob}(\mathcal{B})$$

and morphisms to morphisms in a way that preserves identities and compositions. The functor $F$ is **covariant** if

$$F : A \xrightarrow{f} A' \in \text{Hom}(A, A') \quad \implies \quad F(A) \xrightarrow{F(f)} F(A') \in \text{Hom}(F(A), F(A')),$$

and **contravariant** if

$$F : A \xrightarrow{f} A' \in \text{Hom}(A, A') \quad \implies \quad F(A') \xrightarrow{F(f)} F(A) \in \text{Hom}(F(A'), F(A)).$$

**Exercise 3.4.** For a commutative ring $R$, show that the set of $R$-modules and $R$-module homomorphisms is a category, and that if we fix a module $N$, then

(a) $\text{Hom}_R(N, \bullet)$ is covariant.
(b) \( \text{Hom}_R(\bullet, N) \) is contravariant.

(c) \( \bullet \otimes_R (N) \) is covariant.

(d) Localization is covariant.

What happens when composing functors?

Functors are most interesting when they provide a way to translate between categories that are of very different flavors.

**Example 3.5.** Homology is a functor for any of the homology theories–simplicial, singular, Čech, de Rham–introduced in Chapters 5 and 6.

- if \( X \xrightarrow{1_X} X \) then \( H_i(X) \xrightarrow{H(1_X)} H_i(X) \).
- if \( X \xrightarrow{f} Y \) and if \( Y \xrightarrow{g} Z \) then \( H_i(g \circ f) = H_i(g) \circ H_i(f) \).

**Exercise 3.6.** Connect the two concepts below using category theory.

(a) In Definition 1.9 of Chapter 4, we defined a homotopy of maps: two continuous maps of topological spaces \( f_0, f_1 : S \to T \) are *homotopic* if there is a continuous map \( F \) such that

\[
S \times [0, 1] \xrightarrow{F} T
\]

with \( F(x, 0) = f_0(x) \) and \( F(x, 1) = f_1(x) \); \( F \) deforms \( f_0 \) to the map \( f_1 \).

(b) In Definition 1.7 of Chapter 6, we defined a chain homotopy of complexes: given chain maps

\[
A_i \xrightarrow{\alpha_i} B_i \text{ and } A_i \xrightarrow{\beta_i} B_i
\]

a chain homotopy is a map

\[
A_i \xrightarrow{\gamma_i} B_{i+1}
\]

such that

\[
\alpha_i - \beta_i = \gamma_{i-1} \delta_i + \delta_{i+1} \gamma_i
\]

in the diagram below

\[
\begin{array}{ccc}
\cdots & \xrightarrow{\delta_{i+1}} & A_i & \xrightarrow{\delta_i} & A_{i-1} & \cdots \\
& & \gamma_i & & \alpha_i - \beta_i & \gamma_{i-1} \\
\cdots & \xrightarrow{\delta_{i+1}} & B_i & \xrightarrow{\delta_i} & B_{i-1} & \cdots
\end{array}
\]

Theorem 1.8 of Chapter 6 showed that chain homotopic maps induce the same map on homology. What can you say if the maps in homology in (b) are induced by maps on spaces of type (a)?
4. Persistent Homology

In this section we tackle persistent homology. The focus of our interest will be on the Čech or Rips complex obtained from point cloud data input. Recall from §1 that this is a filtered simplicial complex, varying with a parameter $\epsilon \in \mathbb{R}$.

**Example 4.1.** Consider the filtered simplicial complex below, taken from [53]. The ranks of the homology are depicted as barcodes. There are 7 “snapshots” of the corresponding simplicial complex $\Delta_\epsilon$.

For example, at the time the 3rd snapshot is taken, the simplicial complex $\Delta_\epsilon$ has a single connected component, which is reflected in the fact that for the corresponding value of $\epsilon$, there is a single long bar for $H_0$. Similarly, at the time of the sixth snapshot, all the one and two dimensional holes in $\Delta_\epsilon$ have been filled in, which is reflected that there are no classes surviving (no bars) for $H_1$ and $H_2$.

**4.1. History.** The idea of extracting meaning from points sampled from a space reaches back to the dawn of topology—at least to [80], and in various forms even earlier. A major hurdle was the difficulty (or perhaps, impossibility) of effective implementation; this became surmountable with the advent of high speed computing. The modern development of the theory has roots in work of Frosini [49], [50] and Frosini-Landi [51] on connected components in the early 1990’s, and was extended to higher homology by Robins [86] and Edelsbrunner-Letscher-Zomorodian [42]. From there the theory of persistent homology took off in leaps and bounds; a nice synopsis of the story up to 2008 appears in the survey of Edelsbrunner-Harer [43]. Carlsson-Zomorodian introduced multidimensional/multiparameter persistent homology (MPH) in [23], and we tackle this in the next chapter.
4.2. Persistent Homology and the Bar Code. As discussed at the beginning of this chapter, the starting point for persistent homology is a point cloud dataset $X$, from which we build a filtered topological space

$$X_\epsilon = \bigcup_{p \in X} N_\epsilon(p),$$

where $N_\epsilon(p)$ denotes an $\epsilon$ ball around $p$.

The salient feature is that as in Morse theory, the topology of the spaces $X_\epsilon$ and $\Delta_\epsilon$ (for both the Čech and Rips complexes) only changes at a finite number of values of $\epsilon$. For $\epsilon < \epsilon'$, the inclusions

$$X_\epsilon \hookrightarrow X_{\epsilon'} \quad \text{and} \quad \Delta_\epsilon \hookrightarrow \Delta_{\epsilon'}$$

induce maps in homology.

**Definition 4.2.** The $p^{th}$ persistent homology is the image of the homomorphism

$$H_p(\Delta_\epsilon) \xrightarrow{f_{\epsilon, \epsilon'}} H_p(\Delta_{\epsilon'})$$

When $\epsilon' - \epsilon$ is small, these maps are uninteresting, save at the finite number of parameter values where a new simplex is added in. The finiteness aspect means that the problem can be discretized (see, for example, Carlsson-Zomorodian [22]): reindex $\Delta$ using $\mathbb{Z}$ as an index set, with an inclusion

$$\Delta(i) \hookrightarrow \Delta(i + 1)$$

when a new simplex is added. For a fixed index $i$, there is the usual simplicial boundary map on the chain complex $C_*(\Delta(i))$. Using $R = \mathbb{K}[x]$ with $\mathbb{K}$ a field as the coefficient ring allows us to encode the passage from $\Delta(i)$ to $\Delta(i + 1)$ by multiplication by $x$. This yields a double complex of graded modules, which are discussed in detail in Chapter 8. For a fixed value $i$

$$\cdots \longrightarrow C_{j+1}(\Delta(i), R) \xrightarrow{\partial_{j+1}} C_j(\Delta(i), R) \longrightarrow \cdots$$

is a complex of modules over a principal ideal domain, and the structure theorem of Chapter 3 gives a complete description of the homology.

**Definition 4.3.** A homology class $\alpha$ that first appears in $\Delta(i)$ is said to be born at time $i$. If it does not vanish at a later time then it persists, and corresponds to the free module $x^i \cdot R$, depicted as a long bar as in Example 4.1. If $\alpha$ first vanishes (dies) in $\Delta(i + j)$ then the corresponding module is $x^i \cdot R / \langle x^{i+j} \rangle$, depicted in Example 4.1 as a short bar. The barcode of a point cloud data set $X$ is a graphic representation of PH of a filtered simplicial complex. It is a complete invariant: the homology is equivalent to the barcode, as we saw in Example 4.6 of Chapter 3.
4.3. **Persistence Diagrams and Stability.** What happens if we have two sets of point cloud data \( X \) and \( Y \) that are “near” each other? For persistent homology to be useful, it should have some stability: if \( d(X, Y) \) is small, the persistent homology of \( X \) and \( Y \) should be close, so we need a metric on persistent homology. The **Hausdorff metric** is a standard way to make the set of compact subsets of a metric space into a metric space itself.

**Definition 4.4.** Let \( X \) and \( Y \) be two non-empty subsets of a metric space with metric \( d \). The Hausdorff distance between \( X \) and \( Y \) is

\[
d_H(X, Y) = \max\{\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)\}
\]

Put simply, the Hausdorff distance is the longest distance between a point \( x \in X \) and the point in \( Y \) closest to \( x \). For persistent homology, we will be comparing sets of points \( X \) and \( Y \) such that \( |X| = |Y| \). For the 2-dimensional (births and deaths of homology classes) data of persistent homology, we have the **bottleneck distance**:

**Definition 4.5.** For \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \), the sup norm or \( L_\infty \) distance between \( p \) and \( q \) is

\[
||p - q||_\infty = \max_i |p_i - q_i|.
\]

Let \( \Gamma \) be the set of all bijections from \( X \) to \( Y \). The bottleneck distance between \( X \) and \( Y \) is

\[
d_B(X, Y) = \inf_{\gamma \in \Gamma} \sup_{x \in X} ||x - \gamma(x)||_\infty.
\]

In [31], Cohen-Steiner-Edelsbrunner-Harer prove a stability theorem for persistent homology; [40] calls this result “arguably the most powerful theorem in TDA”. The stability theorem has a Morse theoretic flavor and is phrased in terms of persistence diagrams, which are a different but equivalent way of packaging the 2D data of persistent homology. Here is our roadmap to understand stability.

(a) We first connect the barcode the the persistence diagram. Much of the literature on stability uses persistence diagrams, so we begin by setting up the dictionary. Chapter 7 of [44] gives a detailed description of persistence diagrams and stability. The depiction of persistence diagrams below differs a bit from [44], we add a line at infinity \( \mathbb{R}_\infty^1 \). This is similar to the construction of the projective plane in Chapter 2, and aids us in visualizing the relationship between barcodes and persistence diagrams.

(b) Suppose \( X \) is a manifold and \( f, g \) are Morse functions. By Theorem 2.7 there are a finite number of critical points, and the main result on stability specializes in this context to the statement that the bottleneck distance between the persistence diagrams of \( f \) and \( g \) is bounded by the \( L_\infty \) distance between the functions \( f \) and \( g \) themselves.
(c) The proof by Cohen-Steiner-Edelsbrunner-Harer in [31] involves a fairly intricate diagram chase. We instead give the proof due to Cohen-Steiner-Edelsbrunner-Morozov [32] of a slightly weaker version: Theorem 4.12 applies in the setting of simplicial complexes and monotone functions. This result can be applied in the context of the Čech or Rips complex arising from point cloud data.

(d) In §4.4 we discuss interleaving. Definition 4.9 below involves choosing a set of points that braids or interlaces with the critical points of a function. This will be key in defining when persistence diagrams are close together.

(e) In §4.5 we illustrate these tools in action, sketching results of [77] applying stability to merge trees.

**Definition 4.6.** A persistence diagram is a collection of points, each having an integer multiplicity, in $\mathbb{R}^2 \cup \mathbb{R}^1_{\infty}$, obtained from persistent homology as follows. The multiplicity $\mu_{i,j}^p$ is the number of classes in $H_p$ which are born at time $i$ and die at time $j$; assign $\mu_{i,j}^p$ to the point $(i, j)$. Similarly, $\mu_{i,\infty}^p$ is the number of classes in $H_p$ which are born at time $i$ and persist; assign $\mu_{i,\infty}^p$ to the point $i \in \mathbb{R}^1_{\infty}$. Pictorially, we superimpose $\mathbb{R}^1_{\infty}$ on $\mathbb{R}^2$ at a $y$ value beyond the finite death times.

Thus, a point in the persistence diagram corresponds to a bar in the barcode, and the multiplicity of the point is the number of copies of the bar. For $j \neq \infty$, the distance from a point $(i, j)$ in the persistence diagram to the diagonal line $x = y$ is $j - i$, which is the length of the corresponding bar. Points of the form $(i, \infty)$ correspond to the long bars in the barcode which are born at time $i$. An example will help clarify things.

**Example 4.7.** For the $H_1$ barcode of Example 4.1, the persistence diagram is below, with the horizontal time corresponding to the seven snapshots.

The longest $H_1$ bar is born just after the third snapshot is taken, and dies just before the sixth snapshot, and corresponds to the point $(3.1, 5.9)$. There are no persistent classes in $H_1$ for Example 4.1, hence no points on the line $\mathbb{R}^1_{\infty}$. 
Definition 4.8. A continuous real valued function \( f \) on a space \( X \) has a homological critical value at \( q \) if the map induced by inclusion

\[
\lim_{\epsilon \to 0} [H_p(f^{-1}(-\infty, q - \epsilon)) \to H_p(f^{-1}(-\infty, q + \epsilon))]
\]

is not an isomorphism. The function \( f \) is tame if it has a finite number of homological critical values, and the homology \( H_p(f^{-1}(-\infty, q], \mathbb{K}) \) of the sublevel sets is finite dimensional for all points \( q \in X \).

For example, if \( X \) is smooth and \( f \) is a Morse function then the homological critical points are the usual critical points of \( f \). Tame functions are the right setting for the general stability result of [31], which we state (but don’t prove) as Theorem 4.13.

Definition 4.9. For a tame function \( f \) as in Definition 4.8, let \( \{a_1 < \ldots < a_\ell\} \) be the homological critical values of \( f \) and set \( a_0 = -\infty \) and \( a_{\ell + 1} = +\infty \). For \( i \in \{0, \ldots, \ell\} \) choose interleaving points \( b_i \) such that \( a_i < b_i < a_{i+1} \). Recall from Definition 4.2 that the \( p^{th} \) persistent homology is the rank of the map \( f_{ij}^p \), and write \( \beta_{ij}^p \) for \( \text{rank}(\text{im}(f_{ij}^p)) \). For a pair \( (a_i, a_j) \) with \( 0 \leq i < j \leq \ell + 1 \), define the \( p \)-multiplicity of the pair as

\[
\mu_{ij}^p = (\beta_{ij}^p - \beta_{i+1,j}^p) - (\beta_{i-1,j}^p - \beta_{i-1,j+1}^p).
\]

The left hand bracketed expression measures classes born at time \( i \) or earlier, which die at time \( j \), while the right bracketed expression measures classes born at time \( i - 1 \) or before, and dying at time \( j \). Hence the difference of the two gives the number of classes born at time \( i \), and dying at time \( j \). For a tame function \( f \), let \( D_p(f) \) be the persistence diagram as in Definition 4.6.

Exercise 4.10. Prove that

\[
\beta_{ij}^p = \sum_{a_i \leq r \leq s < a_j} \mu_{ij}^p.
\]

Hint: show \( \beta_{ij}^p \) is the number of points in the “Northwest quadrant” defined by \( \{(x, y) \mid x \leq a_r, y \geq a_s\} \) in the persistence diagram for \( H_p \).

We’re now ready to prove the result of [32]. Let \( f : X \to \mathbb{R} \) be a continuous function on a simplicial complex \( K \), and order the faces of \( K \) so that \( \tau \leq \sigma \) if \( \tau \subseteq \sigma \). Define a piecewise constant monotone approximation \( \overline{f} \) to \( f \) such that

\[
\overline{f}(\tau) \leq \overline{f}(\sigma) \text{ if } \tau \leq \sigma.
\]

One way to do this is via \( \sigma \mapsto \max\{f(x) \mid x \in \sigma\} \).

Definition 4.11. For functions \( f, g : X \to \mathbb{R} \), the sup or \( L_\infty \) norm is

\[
\|f - g\|_\infty = \max_{x \in X} |f(x) - g(x)|.
\]
Theorem 4.12. For monotone $\overline{f}, \overline{g}$ mapping a simplicial complex $K \to \mathbb{R}$,

\[ d_B(D_p(\overline{f}), D_p(\overline{g})) \leq ||\overline{f} - \overline{g}||_{\infty}. \]

Proof. Let $\sigma$ be a face of $K$. Since $\overline{f}$ and $\overline{g}$ are monotone, the homotopy

\[ h_t(\sigma) = (1 - t)\overline{f}(\sigma) + t\overline{g}(\sigma) \text{ for } t \in [0, \ldots, 1] \]

is also monotone; the fact that $h_t$ is monotone means that it induces an ordering on the vertices of $\sigma$; say the ordering changes at $\{t_1, \ldots, t_k\}$ and put $t_0 = 0, t_{k+1} = 1$. For $i \in \{0, \ldots, k\}$, take $r$ and $s$ such that $t_i \leq r < s < t_{i+1}$. If $(\tau, \sigma)$ is a pair of simplices defined for the ordering of vertices that exists for $r \leq t \leq s$, then

\[ (h_r(\tau), h_r(\sigma)) \in D_p(h_r) \text{ and } (h_s(\tau), h_s(\sigma)) \in D_p(h_s). \]

From the definition of the $L_{\infty}$ distance, we then have

\[ d_B(D_p(h_r), D_p(h_s)) \leq ||h_r - h_s||_{\infty} = (s - r) \cdot ||\overline{f} - \overline{g}||_{\infty}. \]

A transposition changes the pairing, but not the persistence diagram, because changing a birth-death pair to a death-birth pair leaves the persistence diagram unchanged. So

\[ d_B(D_p(\overline{f}), D_p(\overline{g})) \leq \sum_{i=0}^{k} d_B(D_p(h_{t_i}), D_p(h_{t_{i+1}})) \]

\[ \leq \left( \sum_{i=0}^{k} (t_{i+1} - t_i) \right) ||\overline{f} - \overline{g}||_{\infty}. \]

Since $\sum_{i=0}^{k} (t_{i+1} - t_i) = 1$, this concludes the proof. \qed

As noted, the proof above is for a somewhat special situation. The stability theorem holds in a more general setting, and we close by giving the statement of the general result of [32].

Theorem 4.13. [Stability Theorem] Let $X$ be a triangulable space and $f, g$ continuous tame real valued functions on $X$. Then the persistence diagrams of $f$ and $g$ satisfy $d_B(D(f), D(g)) \leq ||f - g||_{\infty}$.

The notion of interleaving appearing in Definition 4.9 plays a key role in many versions of stability. We discuss this in the next section.
4. Interleaving. Persistent Homology fits naturally into a category theoretic framework, and this viewpoint will be useful in formulating the concept of *interleaving distance*. The reason to bring interleaving distance into the picture is that it provides a natural way to define a metric on spaces related to diagrams.

**Definition 4.14.** A *generalized* persistence module is a representation of a poset $P$ in a category $\mathcal{C}$. This means that for $x_1 \leq x_2 \leq x_3$ in $P$ there are objects $F(x_i) \in \mathcal{C}$ and morphisms satisfying

$$F(x_1) \xrightarrow{F(\leq_{12})} F(x_2) \xrightarrow{F(\leq_{23})} F(x_3)$$

such that $F(\leq_{11})$ is the identity, and

$$F(\leq_{23}) \cdot F(\leq_{12}) = F(\leq_{13}).$$

This is familiar ground: the conditions are the compatibility conditions we’ve seen defined in several different settings, ranging from direct limit (where the indexing set is again a poset) to the transition functions for vector bundles.

In Exercise 3.6 we revisited homotopies between chain complexes and connected them to homotopies of continuous maps. *Interleaving* is a variation on this theme, introduced by Chazal-Cohen-Steiner-Glisse-Guibas-Oudot in [27]. Roughly speaking, interleaving is a way of sandwiching sequences between each other. As is the case with the “squeeze” theorem on limits encountered in freshman calculus, this means the sequences under consideration are close.

**Definition 4.15.** Let $\mathcal{C}$ be a category and $A_\bullet$, $B_\bullet$ be sets of objects, with maps indexed by $P = (\mathbb{R}, \leq)$. An $\epsilon$-*interleaving* is a sequence of morphisms

\[
\begin{array}{cccc}
\cdots \cdots \to A_t & \xrightarrow{d} & A_{t+\epsilon} & \xrightarrow{d} & A_{t+2\epsilon} & \xrightarrow{d} & A_{t+3\epsilon} & \cdots \\
\phi \downarrow & & \psi \downarrow & & \phi \downarrow & & \psi \\
\cdots \cdots \to B_t & \xrightarrow{\delta} & B_{t+\epsilon} & \xrightarrow{\delta} & B_{t+2\epsilon} & \xrightarrow{\delta} & B_{t+3\epsilon} & \cdots \\
\end{array}
\]

such that the diagrams commute:

$$d^2 = \psi \circ \phi \text{ and } \delta^2 = \phi \circ \psi.$$ 

For example, let $A_t = X_t$ be a filtered topological space, $B_t = \Delta_t$ the associated Čech or Rips complex, with horizontal maps induced by inclusion. The initial version was motivated by TDA, but interleaving fits into a broader categorical framework, explored by Bubenik-de Silva-Scott in [15], [16]. See Lesnick [71] for interleaving in the multiparameter setting, de Silva-Munch-Stefanou [40] for recent categorical work, Bjerkevik-Botnan-Kerber [10], [11] for results on computational complexity of interleaving, and Dey-Fan-Wang [41] for a version where the horizontal maps are not inclusions.
4.5. Vignette: Interleaving and Merge Trees. An illustrative use of interleaving distance is the work of Morozov-Beketayev-Weber [77] on Merge Trees. A merge tree is an encoding of the connected components of the sublevel sets of a function—an edge traces the evolution of a component over time, with two edges meeting in a vertex when the connected components merge.

Definition 4.16. For a function $X \xrightarrow{f} \mathbb{R}$,

- the Reeb graph is the quotient space $X/\sim$ appearing in Definition 1.14 of Chapter 4, with $x_1 \sim x_2$ if they lie in the same connected component of $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$

So for a surface, quotienting contracts every contour line to a point.

- the merge tree $T_f$ is a rooted graph which tracks the connected components of the sublevel sets of $f$. Formally, let

$$\text{Up}(f) = \{(x,y) \in X \times \mathbb{R} \mid y \geq f(x)\}.$$  

Project $\text{Up}(f) \rightarrow \mathbb{R}$ via $\tilde{f}(x,y) = y$; the Reeb graph of $\tilde{f}$ is the merge tree $T_f$. There is a well-defined ancillary map $\hat{f} : T_f \rightarrow \mathbb{R}$ sending $x \in \text{Up}(f) \mapsto y$, where $y$ is the component of $\tilde{f}^{-1}(\tilde{f}(x))$ containing $x$. Note that the projection $\pi$ of a level set of $\tilde{f}$ in $\text{Up}(f)$ to $X$ is a sublevel set of the original function $f$. Let $F_p = \pi(\tilde{f}^{-1}(p)) = f^{-1}(-\infty, p]$.

- for $\epsilon \in \mathbb{R}$, the inclusion $F_p \hookrightarrow F_{p+\epsilon}$ means that a connected component $X$ of $F_p$ is contained in a connected component $Y \subseteq F_{p+\epsilon}$. This allows us to define the $\epsilon$-shift map on the merge tree

$$T_f \xrightarrow{i^\epsilon} T_f$$

as follows: $x \in T_f$ corresponds to a connected component $X$ as above, which maps to $Y$, which corresponds to $y$ in $T_f$; define $i^\epsilon(x) = y$.

The intuition is that $i^\epsilon$ moves $x \in T_f$ towards the root of $T_f$, and stops when it reaches $y$ corresponding to $Y \subseteq F_{p+\epsilon}$ as above.
In order to relate a pair of merge trees $T_f$ and $T_g$, we consider interleaving maps between the two. Continuous maps $T_f \xrightarrow{\alpha^\epsilon} T_g$ and $T_g \xrightarrow{\beta^\epsilon} T_f$

are $\epsilon$-compatible if they satisfy the criteria of Definition 4.15, where

$$\hat{g}(\alpha^\epsilon(x)) = \hat{f}(x) + \epsilon \quad \hat{f}(\beta^\epsilon(y)) = \hat{g}(y) + \epsilon$$
$$\beta^\epsilon \circ \alpha^\epsilon = i^{2\epsilon} \quad \alpha^\epsilon \circ \beta^\epsilon = j^{2\epsilon},$$

with $T_f \xrightarrow{i^{2\epsilon}} T_f$ and $T_g \xrightarrow{j^{2\epsilon}} T_g$ are the $2\epsilon$-shift maps in their trees.

**Definition 4.17.** The interleaving distance on merge trees is given by

$$d_I(T_f, T_g) = \inf \{ \epsilon \mid \text{there exist } \epsilon - \text{compatible maps as above}. \}$$

**Exercise 4.18.** Show that $d_I$ is a metric. See Lemma 1 of [77] if you get stuck. ◇

Here are the two main results of [77] on interleaving distance for merge trees:

**Theorem 4.19.** For $f, g : X \rightarrow \mathbb{R}$, the merge trees $T_f$ and $T_g$ satisfy

$$d_I(T_f, T_g) \leq \| f - g \|_\infty.$$  

**Proof.** We interleave the sublevel sets of $f$ and $g$. If $\| f - g \|_\infty = \epsilon$ then since $F_p = f^{-1}(-\infty, p]$ and $G_q = g^{-1}(-\infty, q]$

$$F_p \subseteq G_{p+\epsilon} \subseteq F_{p+2\epsilon}.$$  

Now,

$$\hat{p} \in T_f \text{ corresponds to a component } X_p \subseteq F_p \subseteq G_{p+\epsilon},$$

so the component $X_p$ is contained in some component $Y_{p+\epsilon}$ of $G_{p+\epsilon}$. If $\hat{q}$ is the point of $T_g$ corresponding to $Y_{p+\epsilon}$, then

$$\alpha^\epsilon(\hat{p}) = \hat{q}.$$  

By construction, if $\hat{f}(\hat{p}) = p$ then $\hat{g}(\alpha^\epsilon(\hat{p})) = p + \epsilon$. The shift maps are induced by inclusions so

$$\beta^\epsilon \circ \alpha^\epsilon = i^{2\epsilon} \quad \text{and} \quad \alpha^\epsilon \circ \beta^\epsilon = j^{2\epsilon}$$

and we conclude that $d_I(T_f, T_g) \leq \epsilon$. □

There is a similar result for bottleneck distance. The proof uses a version of the result of [27], simplified to the setting $p = 0$. Let $f, g : X \rightarrow \mathbb{R}$ be tame functions such that there are $\epsilon$-interleavings as below

$$H_0(F_t) \xrightarrow{\phi^\epsilon} H_0(G_{t+\epsilon}) \text{ and } H_0(G_t) \xrightarrow{\psi^\epsilon} H_0(F_{t+\epsilon}),$$

which commute with the inclusion maps. Then

$$d_B(D_0(f), D_0(g)) \leq \epsilon.$$
Theorem 4.20. For \( f, g : X \rightarrow \mathbb{R} \), the merge trees \( T_f \) and \( T_g \) satisfy
\[
d_B(D_0(f), D_0(g)) \leq d_I(T_f, T_g)
\]

Proof. Because merge trees contract sublevel sets to points and \( H_0 \) measures the number of connected components,
\[
X \xrightarrow{f} \mathbb{R} \text{ and } T_f \xrightarrow{\hat{f}} \mathbb{R}
\]
have the same zero-dimensional persistence diagrams
\[
D_0(f) = D_0(\hat{f}).
\]
Hence it suffices to show that
\[
d_B(D_0(\hat{f}), D_0(\hat{g})) \leq d_I(T_f, T_g) = \epsilon.
\]
Since \( d_I(T_f, T_g) = \epsilon \), we have maps
\[
T_f \xrightarrow{\alpha^*} T_g \text{ and } T_g \xrightarrow{\beta^*} T_f
\]
which commute with the induced inclusion maps. So in particular, \( \hat{f}^{-1}(-\infty, a] \) and \( \hat{g}^{-1}(-\infty, a] \) are \( \epsilon \)-interleaved, hence so are the corresponding groups
\[
H_0(\hat{f}^{-1}(-\infty, a]) \text{ and } H_0(\hat{g}^{-1}(-\infty, a]).
\]
Applying the variant of [27] described above concludes the proof. \( \square \)
Chapter 8

Multiparameter Persistent Homology

In [23], Carlsson and Zomorodian introduced an extension of persistent homology to the setting of filtrations depending on more than one parameter. Near then end of [23], they write

Our study of multigraded objects shows that no complete discrete invariant exists for multidimensional persistence. We still desire a discriminating invariant that captures persistent information, that is, homology classes with large persistence.

In this chapter we use tools of multigraded algebra to explore multiparameter persistent homology (MPH). While there are natural analogs of the free summands (long bars) and torsion components (short bars) of persistent homology, a new phenomenon arises. In particular, there are intermediate components, which are not full dimensional (not long bars), but do not die in high degree (not short bars).

It turns out that the associated primes introduced in Chapter 4 provide a useful tool for analyzing multiparameter persistent homology. Just as persistent homology is a \( \mathbb{Z} \)-graded module over \( \mathbb{K}[x] \), multiparameter persistent homology with \( n \)-parameters is a \( \mathbb{Z}^n \)-graded module over \( \mathbb{K}[x_1, \ldots, x_n] \). This places very strong constraints on the associated primes. Roughly speaking, it allows us to visualize components of an MPH module as translates of coordinate subspaces.

- Definition and Examples.
- Graded algebra, Hilbert function, series, polynomial.
- Associated Primes and \( \mathbb{Z}^n \)-graded modules.
- Filtrations and Ext.
1. Definition and Examples

In this section we define multiparameter persistent homology; while the definitions are a bit lengthy the thing to keep in mind is that the constructions are simply the $\mathbb{Z}^n$-graded analogs of persistent homology. We illustrate this by working thru an example from [23] in detail. The subsequent sections introduce the tools from multigraded algebra used in [60] to analyze multiparameter persistent homology.

1.1. Multiparameter persistence. This section develops MPH, following [23].

**Definition 1.1.** Denote by $\mathbb{Z}^n$ the set of $n$-tuples of natural numbers, and define the following partial order on $\mathbb{Z}^n$: for any pair of elements $u, v \in \mathbb{Z}^n$ we define $u \preceq v$ iff $u_i \leq v_i$ for all $i = 1, \ldots, n$, where we write $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. Given a collection of simplicial complexes $\{\Delta_u\}_{u \in \mathbb{Z}^n}$ indexed by $\mathbb{Z}^n$, we say that $\{\Delta_u\}_{u \in \mathbb{Z}^n}$ is an $n$-filtration if whenever $u \preceq v$ we have that $\Delta_u \subseteq \Delta_v$. If there exists $u' \in \mathbb{Z}^n$ such that $\Delta_u = \Delta_{u'}$ for all $u \succeq u'$, then we say that the $n$-filtration stabilizes. A multifiltration is an $n$-filtration for some $n$.

An $n$-filtered simplicial complex is a simplicial complex $\Delta$ together with a multifiltration $\{\Delta_u\}_{u \in \mathbb{Z}^n}$ that stabilizes and such that $\Delta = \cup_{u \in \mathbb{Z}^n} \Delta_u$. An $n$-filtered simplicial complex $(\Delta, \{\Delta_u\}_{u \in \mathbb{Z}^n})$ is finite if $\Delta$ is finite. A multifiltered simplicial complex is an $n$-filtered simplicial complex for some $n \geq 1$.

Given a multifiltered simplicial complex $(\Delta, \{\Delta_u\}_{u \in \mathbb{Z}^n})$, for each $x \in \Delta$ we call the minimal elements $u \in \mathbb{Z}^n$ (with respect to the partial order $\preceq$) at which it enters the filtration its entry degrees. If every $x \in \Delta$ has exactly one entry degree we call the multifiltered space one-critical.

Let $\Delta$ be a multifiltered simplicial complex, and let $i = 0, 1, 2, \ldots$. For any $u \in \mathbb{Z}^n$ denote by $C_i(\Delta_u)$ the $\mathbb{K}$-vector space with basis given by the $i$-simplices of $\Delta_u$, and similarly by $H_i(\Delta_u)$ the $i$th simplicial homology with coefficients in $\mathbb{K}$. Whenever $u \preceq v$ we have that the inclusion maps $\Delta_u \to \Delta_v$ induce $\mathbb{K}$-linear maps $\psi_{u,v}: C_i(\Delta_u) \to C_i(\Delta_v)$ and $\phi_{u,v}: H_i(\Delta_u) \to H_i(\Delta_v)$ such that whenever $u \preceq w \preceq v$ we have that $\psi_{w,v} \circ \psi_{u,w} = \psi_{u,v}$, and similarly $\phi_{w,v} \circ \phi_{u,w} = \phi_{u,v}$. We thus give the following definition:

**Definition 1.2.** Let $\Delta$ be a multifiltered simplicial complex. The $i$th-chain module of $\Delta$ over $\mathbb{K}$ is the tuple

$$\{C_i(\Delta_u)\}_{u \in \mathbb{Z}^n}, \{\psi_{u,v}: C_i(\Delta_u) \to C_i(\Delta_v)\}_{u \preceq v}.$$

Similarly, the simplicial homology with coefficients in $\mathbb{K}$ of $\Delta$ is the tuple

$$\{H_i(\Delta_u)\}_{u \in \mathbb{Z}^n}, \{\phi_{u,v}: H_i(\Delta_u) \to H_i(\Delta_v)\}_{u \preceq v},$$

where the maps $\psi_{u,v}$ and $\phi_{u,v}$ are those induced by the inclusions.
Definition 1.3. An \( n \)-parameter persistence module (or multiparameter persistence module) is defined by an \( n \)-tuple \( (\{ M_u \}_{u \in \mathbb{Z}^n}, \{ \phi_{u,v} : M_u \rightarrow M_v \}_{u \preceq v}) \) where \( M_u \) is a \( \mathbb{K} \)-module for each \( u \) and \( \phi_{u,v} \) is a \( \mathbb{K} \)-linear map, such that whenever \( u \preceq w \preceq v \) we have \( \phi_{w,v} \circ \phi_{u,w} = \phi_{u,v} \). A morphism of multiparameter persistence modules

\[
f : (\{ M_u \}_{u \in \mathbb{Z}^n}, \{ \phi_{u,v} \}_{u \preceq v}) \rightarrow (\{ M'_u \}_{u \in \mathbb{Z}^n}, \{ \phi'_{u,v} \}_{u \preceq v})
\]

is a collection of \( \mathbb{K} \)-linear maps \( \{ f_u : M_u \rightarrow M'_u \}_{u \in \mathbb{Z}^n} \) such that \( f_v \circ \phi_{u,v} = \phi'_{u,v} \circ f_u \) for all \( u \preceq v \).

Let \( \Delta \) be a multifiltered simplicial complex. An example of a morphism of MPH modules is given by the differentials of the simplicial chain complex

\[
C_\bullet(\Delta_u) : C_i(\Delta_u) \xrightarrow{d_i} C_{i-1}(\Delta_u),
\]

for each \( u \in \mathbb{Z}^n \), which induce morphisms of multiparameter persistence modules

\[
(\{ C_i(\Delta_u) \}_{u \in \mathbb{Z}^n}, \{ \psi_{u,v} \}_{u \preceq v}) \rightarrow (\{ C_{i-1}(\Delta_u) \}_{u \in \mathbb{Z}^n}, \{ \psi_{u,v} \}_{u \preceq v})
\]

for any \( i \geq 0 \) and where \( C_{-1}(\Delta_u) = 0 \). Notice that in the spirit of Chapter 7 it is also possible to define an \( n \)-parameter persistence module as a functor from the poset category \( \mathbb{Z}^n \) to the category with objects \( \mathbb{K} \)-vector spaces and morphisms \( \mathbb{K} \)-linear maps. Indeed, in [23] Carlsson-Zomorodian show the categories of MPH modules and fine-graded modules are isomorphic.

Example 1.4. In [20], Carlsson, Singh and Zomorodian analyze the simplicial homology of the one-critical bifiltration in Fig. 1.

![Figure 1](image-url)  

The differentials in the multifiltered simplicial chain complex are given by
8. Multiparameter Persistent Homology

\[
d_1 = \begin{bmatrix}
-x_2 & -x_2 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2^2 & 0 & -x_1^2x_2 & -x_2^2 & 0 & 0 & 0 \\
0 & 0 & x_1^2x_2 & 0 & -x_1 & -x_2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & x_2 & x_1 & -x_1^2 \\
0 & x_1x_2^2 & 0 & x_2^2 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
d_2 = \begin{bmatrix}
x_1 & 0 \\
-x_1 & 0 \\
0 & 0 \\
-x_1^2 & 0 \\
0 & x_1^2x_2 \\
0 & -x_1^3 \\
0 & x_1^3x_2 \\
0 & 0
\end{bmatrix},
\]

where the bases of 0, 1 and 2-simplices are ordered lexicographically. For example, the first column of \(d_1\) reflects the fact that the edge \([ab]\) appears at position \((1, 2)\), so has bidegree \(x_1x_2^2\), while \([a]\) appears in position \((1, 1)\) so has bidegree \(x_1x_2\), and \([b]\) appears in position \((0, 0)\). To preserve the grading, we need to decorate the usual boundary \(\partial_1([ab]) = [b] - [a]\) with coefficients reflecting degree:

\[
d_1([ab]) = x_1x_2^2[b] - x_2[a].
\]

An easy calculation shows that \(\ker(d_1)\) is generated by the rows of

\[
L = \begin{bmatrix}
0 & 0 & 0 & 0 & -x_2 & x_1 & x_2 & 0 \\
-1 & 1 & 0 & -x_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -x_1^2 & x_1x_2^2 & 0 & x_1x_2^2 & x_2^2
\end{bmatrix},
\]

There are two obvious relations

\[
x_1^2 \cdot \text{row}_1(L) = \text{col}_2(d_2)
\]

\[
x_1 \cdot \text{row}_2(L) = \text{col}_1(d_2)
\]

In fact, these are the only relations, so \(H_1(\Delta)\) has minimal presentation

\[
R(-3, -1) \oplus R(-2, -2) \rightarrow R(-1, -1) \oplus R(-1, -2) \oplus R(-2, -2).
\]

The shifts in the terms \(R(*, \ast)\) reflect the grading on \(R = \mathbb{K}[x_1, x_2]\), which we discuss in detail in the next section. For now, note that the columns of \(d_1\) are graded by the degrees of the edges: for example the last column corresponds to edge \([ef]\), which appears in degree \((2, 0)\). The last row of \(L\) corresponds to a generator for \(H_1\), and when multiplied against \(d_1\) has all entries of degree \((2, 2)\).
2. Graded algebra, Hilbert function, series, polynomial

In certain situations, graded algebra turns complicated questions about algebraic objects into questions about vector spaces. This section covers the main invariants of graded modules: the Hilbert function and the Hilbert series, as well as how to read off a first measure of the size of a module—\textit{the rank}. For a module \(M\) over a PID, the rank of \(M\) is the rank of the free summand, so corresponds in TDA to the number of long bars. In particular, the rank of an MPH module is a natural choice of proxy for the number of long bars.

**Definition 2.1.** Let \(R\) be an integral domain, \(\mathbb{K} = R_{(0)}\) the field of fractions as in Chapter 2, and \(M\) a finitely generated \(R\)-module. The \textit{rank} of \(M\) is

\[
\text{rank}(M) = \dim_{\mathbb{K}}(M \otimes_R \mathbb{K}) = \dim_{\mathbb{K}}(M_{(0)})
\]

**Example 2.2.** The \(\mathbb{Z}\)-module \(\mathbb{Z}^m \bigoplus_{i=1}^n \mathbb{Z}/q_i\) has rank \(m\).

**Definition 2.3.** Let \(G\) be an abelian group. A ring \(R\) is \(G\)-graded if

\[
R = \bigoplus_{g \in G} R_g,
\]

where if \(r_i \in R_i\) and \(r_j \in R_j\) then

\[
 r_i \cdot r_j \in R_{i+j}.
\]

Note that each \(R_i\) is itself a module over \(R_0\). A graded module \(M\) over \(R\) is a module with a direct sum decomposition

\[
M = \bigoplus_{g \in G} M_g,
\]

where if \(r_i \in R_i\) and \(m_j \in M_j\) then

\[
 r_i \cdot m_j \in M_{i+j}.
\]

For applications in topological data analysis, the case of interest is \(G = \mathbb{Z}^n\). For these modules, we can determine the rank of a module from the growth rate of the Hilbert Function.

**Example 2.4.** A polynomial ring \(R = \mathbb{K}[x_1, \ldots, x_n]\) is \(\mathbb{Z}\)-graded. The component \(R_i\) is the \(\mathbb{K}\)-vector space of polynomials \(f(x_1, \ldots, x_n)\) such that every monomial appearing \(f\) is of the same degree \(i\), as in Exercise 4.16 of Chapter 2. A simple induction on the number of variables shows that

\[
\dim_{\mathbb{K}} R_i = \binom{n - 1 + i}{i}.
\]
2.1. The Hilbert function. For a $G$-graded module $M$ over a $G$-graded ring $R$ with $R_0 = \mathbb{K}$ a field, the Hilbert function of $M$ records the dimensions of the vector spaces $M_g, g \in G$. Henceforth, we restrict to the case $G = \mathbb{Z}^n$.

Definition 2.5. Let $M$ be an $\mathbb{Z}^n$-graded module over $R = \mathbb{K}[x_1, \ldots, x_n]$. The Hilbert function of $M$ is the function $HF(M, u) = \dim_\mathbb{K} M_u$.

Example 2.6. For the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$, the two most common gradings are the $\mathbb{Z}$-grading appearing in Example 2.4, and the fine or $\mathbb{Z}^n$ grading. In the fine grading with $u = (u_1, \ldots, u_n)$, $R_u$ is one dimensional with basis $x^u$. For $1 \leq r \leq n$ there are other $\mathbb{Z}^r$-gradings possible. For example, consider the ring $T = \mathbb{K}[x_1, \ldots, x_4]$, where $\{x_1, x_2\}$ have degree $(1,0)$ and $\{x_3, x_4\}$ have degree $(0,1)$. This is a $\mathbb{Z}^2$-grading (or bigrading). The bidegree $(1,2)$ component of $T$ is

$$T_{(1,2)} = \text{Span}\{x_1x_3^2, x_1x_3x_4, x_1x_4^2, x_2x_3^2, x_2x_3x_4, x_2x_4^2\}$$

A $\mathbb{Z}^n$-graded Hilbert function is an $n$-dimensional array, with the $u = (u_1, \ldots, u_n)$ entry equal to the dimension of $M_u$. In the fine grading, the Hilbert function of $R$ itself is an $n$-dimensional array with a one in every position which has all indices non-negative, and zeroes elsewhere. The Hilbert function of $R(-u)$ is the same array, but with the origin translated to position $u$; this process is known as shifting.

Definition 2.7. For a finitely generated $R$-module $M$, a free resolution is an exact sequence of free modules terminating in $M$:

$$\cdots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \xrightarrow{} 0.$$  

When working in the category of graded rings and graded modules, maps must preserve the graded structure: $\phi : M \rightarrow N$ takes $M_u \rightarrow N_{u}$.

Example 2.8. In Example 4.17 of Chapter 2, we considered the ideal $I$ of a $2 \times 2$ minors of a $2 \times 3$ matrix $A = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}$. If we grade $R$ by $\mathbb{Z}$, then a free $\mathbb{Z}$-graded resolution for $M = R/I$ is given by

$$0 \xrightarrow{} R(-3)^2 \xrightarrow{\begin{bmatrix} x & y \\ -y & -z \end{bmatrix}} R(-2)^3 \xrightarrow{\begin{bmatrix} xz-y^2 & xw-yz & yw-z^2 \end{bmatrix}} R \xrightarrow{} R/I$$

Elements of the kernel of $d_i$ are called $i^{th}$ syzygies. In the special situation above, the first syzygies arise by stacking $[x,y,z]$ or $[y,z,w]$ as a third row of $A$; the determinant of the resulting $3 \times 3$ matrix has a repeat row, and hence yields a syzygy. The leftmost matrix has no kernel so the process stops. Since the map

$$F_1 = R^3 \xrightarrow{d_1} R^3 = F_0$$

is multiplication by quadrics, to preserve the $\mathbb{Z}$-grading, we must shift so that the degrees of the generators of $F_1$ have degree 2. Similarly, as the leftmost map $d_1$ has degree one entries and maps onto a target generated in degree two, the generators of $F_2$ are in degree three.
In nice situations, free resolutions are actually finite; for a proof see [47].

**Theorem 2.9.** [Hilbert Syzygy Theorem] A finitely generated $R = \mathbb{K}[x_1, \ldots, x_n]$ module $M$ has a free resolution of length at most $n$.

**Lemma 2.10.** For a finitely generated graded $R = \mathbb{K}[x_1, \ldots, x_n]$-module $M$ with free resolution $F_\bullet$ as in Definition 2.7,

- (a) $HF(M, u) = \sum_{i \geq 0} (-1)^i HF(F_i, u)$
- (b) $\text{rank}_{R}(M) = \sum_{i \geq 0} (-1)^i \text{rank}_{R}(F_i)$.

**Proof.** The key point is that because the maps in a graded free resolution preserve the grading, the free resolution is exact in each graded degree. Since for each degree $u$, the degree $u$ components of $F_\bullet$ are an exact sequence of vector spaces over $\mathbb{K}$, so (a) follows from Example 2.4 of Chapter 5. For part (b), by Theorem 2.9 $M$ has a finite free resolution. Tensoring the (finite) exact sequence of Definition 2.7 with the field of fractions of $R$ is equivalent to localizing. By Theorem 3.3 of Chapter 2 localization is exact, so (b) follows from Example 2.4 of Chapter 5. □

**Lemma 2.11.** For a finitely generated $\mathbb{Z}^n$-graded $R = \mathbb{K}[x_1, \ldots, x_n]$-module $M$, $\text{rank}_{R}(M) = HF(M, u)$ for $u \gg 0$.

**Proof.** A finitely generated, $\mathbb{Z}^n$-graded free module $F$ is of the form

$$F = \bigoplus_{j=1}^{m} R(-u_j), \text{ with } u_j \in \mathbb{Z}^n.$$  

Since $HF(R(-u_j), v) = 1$ if $v \geq u_j$ and 0 otherwise, when $v \geq u_j$ for all $j \in \{1, \ldots, n\}$

$$HF(F, v) = n = \text{rank}_{R}(F).$$

By Theorem 2.9 $M$ has a finite free resolution; apply Lemma 2.10. □

**Example 2.12.** [Hilbert Polynomial] In the special case of a $\mathbb{Z}$-graded ring $R$ as in Example 2.8, the Hilbert function of a graded $R$-module $M$ is given by a polynomial when $i \in \mathbb{Z}$ is sufficiently large:

$$HF(M, i) = P(M, i) \in \mathbb{Q}[i], \text{ when } i \gg 0.$$ 

This can be proved by induction on the number of variables, using the four term exact sequence

$$0 \rightarrow \ker(\cdot x_n) \rightarrow M(-1) \xrightarrow{x_n} M \rightarrow \text{coker}(\cdot x_n) \rightarrow 0,$$

using that $\ker(\cdot x_n)$ and $\text{coker}(\cdot x_n)$ are $\mathbb{K}[x_1, \ldots, x_{n-1}]$-modules. The polynomial $P(M, i)$ has the form

$$P(M, i) = \frac{a_m}{m!} i^m + \ldots + a_0, \text{ with } m \leq n - 1.$$
In the context of projective geometry with \( M = R/I \) as in Example 2.8, the dimension of \( V(I) \) is the degree of \( P(R/I, i) \), and the degree of \( V(I) \) is the (normalized) lead coefficient \( a_m \). The dimension of \( V(I) \) is the number of times it can be cut with a general hyperplane until we’re left with a set of points; the number of points (with multiplicity, if \( V(I) \) is singular) is the degree of \( V(I) \). By Example 2.4

\[
P(R(-j), i) = \binom{n - 1 + i - j}{n - 1}
\]

**Exercise 2.13.** Use the free resolution which appears in Example 2.8 to show that \( P(R/I, i) = 3i + 1 \). Thus the degree of \( V(I) \) is three, and the dimension is one. The corresponding curve in \( \mathbb{P}^3 \) is known as the **twisted cubic**.

### 2.2. The Hilbert series.

The Hilbert series provides a compact way of packaging the data of the Hilbert function into a formal power series.

**Definition 2.14.** For a finitely generated \( R = \mathbb{K}[x_1, \ldots, x_n] \)-module \( M \) which is \( \mathbb{Z}^n \)-graded, the **multigraded Hilbert series** of \( M \) is the formal power series in \( \mathbb{Z}[\lbrack t_1, \ldots, t_n \rbrack] \) defined as follows:

\[
HS(M, t) = \sum_{\mathbf{u} \in \mathbb{Z}^n} HF(M, \mathbf{u}) t^\mathbf{u}.
\]

Inducting on the number of variables yields

\[
(2.1) \quad HS(R(-\mathbf{u}), t) = \frac{t^\mathbf{u}}{\prod_{i=1}^{n} (1 - t_i)}.
\]

By Lemma 2.10 the Hilbert function is additive on exact sequences, so this is also true for the Hilbert series. Hence applying Theorem 2.9 we have

\[
(2.2) \quad HS(M, t) = \sum_{i=0}^{n} (-1)^i HS(F_i, t).
\]

For a finitely generated, multigraded \( R \)-module \( M \), it follows from the Hilbert Syzygy Theorem and Equations (2.1)–(2.2) that the multigraded Hilbert series of \( M \) is a rational polynomial of the form

\[
(2.3) \quad HS(M, t) = \frac{P(t_1, \ldots, t_n)}{\prod_{i=1}^{n} (1 - t_i)}.
\]

The polynomial \( P(t_1, \ldots, t_n) \) in Equation (2.3) is an invariant of the module.

**Lemma 2.15.** For a finitely generated graded \( R = \mathbb{K}[x_1, \ldots, x_n] \)-module \( M \), \( \text{rank}_R(M) \) is equal to \( P(1) \), where \( P \) is as in Equation 2.3.

**Proof.** To see that \( \text{rank}_R(M) \) is the numerator of \( HS(M, t) \) evaluated at 1, note that it holds for free modules by Equation (2.1). Now apply Lemma 2.10. \( \square \)
Example 2.16. Note that if $M$ is a one-parameter persistence module, we have

$$HS(M, t) = \sum_{\text{long bars}} \frac{t^{b_l}}{(1 - t)} + \sum_{\text{short bars}} \frac{(t^{b_s} - t^{d_s})}{(1 - t)}$$

where $b_l$ is the birth time for a long bar, $b_s$ the birth time for a short bar, and $d_s$ the death time of a short bar. So

$$HS(M, 1) = \text{rank}_R(M)$$

is indeed the number of long bars.

We close by revisiting Example 1.4

Example 2.17. In Example 1.4, $H_1(\Delta) = \text{coker}(\delta_1)$, with

$$\delta_1 = \begin{bmatrix} x_1^2 & 0 \\ 0 & x_1 \\ 0 & 0 \end{bmatrix}$$

It is easy to see that $\delta_1$ has no kernel itself, so we have a free resolution for $H_1(\Delta)$

$$0 \longrightarrow F_1 \xrightarrow{\delta_1} F_0 \longrightarrow H_1(\Delta) \longrightarrow 0,$$

so

$$HS(H_1(\Delta), t) = HS(F_0, t) - HS(F_1, t) = \frac{t_1t_2 + t_1t_2^2 + t_2^2}{(1-t_1)(1-t_2)} - \frac{t_1^2t_2 + t_2^2}{(1-t_1)(1-t_2)}.$$

It follows that $P(H_1(\Delta), 1) = 1$, so $\text{rank}(H_1(\Delta)) = 1$. In fact, because in this case the presentation is so simple, we see that

$$H_1(\Delta) \simeq (R(-1, -1)/\langle x_1^2 \rangle) \bigoplus (R(-1, -2)/\langle x_1 \rangle) \bigoplus R(-2, -2).$$

In particular, note that the first two terms

$$(R(-1, -1)/\langle x_1^2 \rangle) \bigoplus (R(-1, -2)/\langle x_1 \rangle)$$

are not free, but also do not vanish in high degree in the $x_2$ variable. For any principal ideal $\langle f \rangle$ with $\text{deg}(f) = u$, we have an exact sequence:

$$0 \longrightarrow R(-u) \xrightarrow{f} R \longrightarrow R/\langle f \rangle \longrightarrow 0,$$

so

$$HS(R/\langle f \rangle, t) = \frac{1 - t^u}{\prod_{i=1}^u (1 - t_i)}.$$

As a module, the summand $R(-1, -2)/\langle x_1 \rangle$ above is supported on $V(x_1)$. In the next section, we will see that the associated primes introduced in Chapter 4 provide a way to understand the structure of MPH modules, which generally do not decompose as a direct sum as they do in this example.

Exercise 2.18. Show $HS(R(-1, -2)/\langle x_1 \rangle, t) = \frac{t_1t_2}{1-t_2}$.
3. Associated Primes and $\mathbb{Z}^n$-graded modules

In §4 of Chapter 4, we defined the associated primes $\text{Ass}(M)$ of a module $M$:

\[ P \in \text{Ass}(M) \iff P = \text{ann}(m) \text{ for some } m \in M. \]

**Definition 3.1.** For a Noetherian ring $R$, the *dimension* (or Krull dimension) is

\[ \dim(R) = \sup_n \{ P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \subsetneq R \} \]

with the $P_i$ prime ideals in $R$.

If $R$ is an integral domain (which will always be the case for us), there is a unique common starting point for chains as above: $0 = P_0$. Note that if $R$ is not Noetherian, then there can be infinite ascending proper chains of ideals.

**Definition 3.2.** Let $R$ be a ring and $I$ an ideal in $R$. The *codimension* of $I$ is

\[ \sup_m \{ P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_m \subseteq I \} \]

**Example 3.3.** For $R = \mathbb{K}[x_1, \ldots, x_n]$, since

\[ \{0 \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \cdots \subsetneq \langle x_1, \ldots, x_n \rangle \} \]

is a proper chain of prime ideals, $\dim(R) \geq n$; it is not hard to show this is an equality. Consider the ideal $I = \langle x_1, \ldots, x_m \rangle \subseteq R$. From the definition, $I$ has codimension at least $m$, and this too is an equality. Geometric mnemonic: $\text{V}(I) \subseteq \mathbb{K}^n$ is a linear space defined by the vanishing of $m$ independent linear forms, so is of dimension $n - m$, hence of complimentary dimension $m$. So we have

\[ \text{codim}(I) = \dim(\text{V}(I)) \subseteq \mathbb{K}^n. \]

A word of caution: If $I$ is defined by homogeneous polynomials and we work projectively in $\mathbb{P}^{n-1}$, then the dimension of $\text{V}(I)$ as a projective variety is one less than the dimension as a variety in $\mathbb{K}^n$. For example $\text{V}(x_0, x_1)$ defines a line in $\mathbb{P}^3$, but $\langle x_0, x_1 \rangle$ is codimension two.

For $\mathbb{Z}$-graded rings and modules as in Example 2.8, we noted that maps also have to preserve gradings; this also applies to $\mathbb{Z}^n$-graded modules.

**Theorem 3.4.** The only $\mathbb{Z}^n$-graded prime ideals in $R = \mathbb{K}[x_1, \ldots, x_n]$ are

\[ (3.1) \quad P = \langle x_{i_0}, \ldots, x_{i_r} \rangle. \]

**Proof.** In the $\mathbb{Z}^n$-grading, the only polynomial of degree $\mathbf{u} = (u_1, \ldots, u_n)$ is

\[ x^\mathbf{u} = x_1^{u_1} \cdots x_n^{u_n} \]

Hence, the only $\mathbb{Z}^n$ homogeneous ideals are ideals generated by monomials. A prime ideal $P$ cannot contain a monomial of degree two or more as a minimal generator, since if $x_1x_2 \in P$ then $x_1 \in P$ or $x_2 \in P$. Hence a prime ideal $P$ is generated by monomials iff $P$ is generated by a subset of the variables. \qed
As an immediate consequence, we have

**Corollary 3.5.** The associated primes of a fine graded module are subsets of variables. Hence an MPH module is supported on a coordinate subspace arrangement.

**Example 3.6.** Modify Example 1.4 by adding vertex $[g]$, edges $[ag], [bg], [fg]$ and triangles $[abg], [bfg], [afg]$ in degree $(1, 3)$. The corresponding $\mathbb{Z}^2$-filtered complex $\Delta'$ is the same as the complex of Example 1.4 in degrees $(a, a)$ with $a \leq 2$, and for $a = 3$ is pictured below.

![Figure 2. The simplices of degree $(*, 3)$ in $\Delta'$](image)

Keeping the ordering of the faces of $\Delta$ from Example 1.4, and adding

$[g]$ as the last ordered basis element for $C_0(\Delta')$

$[ag], [bg], [fg]$ as the last ordered basis elements for $C_1(\Delta')$

$[abg], [afg], [bfg]$ as the last ordered basis elements for $C_2(\Delta')$

we have

$$ d_2(\Delta') = \begin{bmatrix} x_1 & 0 & x_2 & 0 & 0 \\ -x_1 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -x_1^2 & 0 & 0 & 0 & x_1 x_2 \\ 0 & x_1^2 x_2 & 0 & 0 & 0 \\ 0 & -x_1^3 & 0 & 0 & 0 \\ 0 & x_1^2 x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} $$

We compute that $\ker(d_2(\Delta'))$ is generated by $[x_2, 0, -x_1, x_1, -x_1]^T$, hence

$$ H_2(\Delta') \cong R(-1, -3). $$

**Exercise 3.7.** Compute $d_1(\Delta')$, and show that

$$ H_1(\Delta') \cong R(-2, -2) \oplus R(-1, -1)/x_1^2 \oplus R(-1, -2)/\langle x_1, x_2 \rangle. $$

The associated primes are $\{0, \langle x_1 \rangle, \langle x_1, x_2 \rangle\}$. \diamond
4. Filtrations and Ext

The examples of MPH modules we’ve encountered so far have all decomposed as direct sums, but this is atypical. In general, our goal will be to stratify or filter the MPH module, to obtain an analog of the barcode. We will use the associated primes to do this, which leads to the first question: in general, how do we identify elements of $\text{Ass}(M)$? There are general algorithms for identifying the associated primes of a graded $R = \mathbb{K}[x_1, \ldots, x_n]$-module. However, these algorithms rely on the use of Gröbner bases [33], and in the worst case Gröbner bases can require doubly exponential runtime. In the setting of multidimensional persistence, Corollary 3.5 shows that an associated prime $P$ of $M$ is a subset of the variables. This places extremely strong constraints on the structure of fine graded modules; in particular, there is a finite subset of possible associated primes.

The algorithm mentioned above for identifying associated primes uses the derived functor $\text{Ext}$, so this is the right time to skip to Chapter 9 and read §1 and §2 if you’re encountering $\text{Ext}$ for the first time. Combining the abstract algebra analysis using $\text{Ext}$ with the strong constraints on the associated primes of a fine graded module will provide insight into the structure of MPH, as well as yielding one candidate for a higher dimensional proxy for the barcode.

**Theorem 4.1.** For a finitely generated graded $R = \mathbb{K}[x_1, \ldots, x_n]$-module $M$, a prime ideal $P$ of codimension $c$ is in $\text{Ass}(M)$ iff it is in $\text{Ass}(\text{Ext}^c(M, R))$.

**Proof.** See [48] for complete details, or [88] for a quick sketch. □

**Example 4.2.** Let $R = \mathbb{K}[x_1, x_2]$ and $M = R/I$ with $I = \langle x_1^2, x_1 x_2 \rangle$. In Exercise 4.9 of Chapter 4, we saw that the associated primes of $I$ are $\{\langle x_1 \rangle, \langle x_1, x_2 \rangle\}$, which followed from the primary decomposition

$$ I = \langle x_1^2, x_2 \rangle \cap \langle x_1 \rangle. $$

Chapter 9 gives a recipe to compute $\text{Ext}^i(M, R)$: take a free resolution $F_\bullet$ for $M$, drop $M$ from the sequence, apply $\text{Hom}_R(\bullet, R)$ to $F_\bullet$, and compute homology. This seems like an arduous task, until we roll up our sleeves and do it.

- **Step 1:** A free resolution for $R/I$ is easy to do by hand:

$$ 0 \longrightarrow R \xrightarrow{\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2 \end{bmatrix}} R \longrightarrow R/I. $$

- **Step 2:** dropping $R/I$ and applying $\text{Hom}_R(\bullet, R)$—which simply transposes the differentials—yields

$$ 0 \longrightarrow R \xrightarrow{\begin{bmatrix} x_1^2 \\ x_1 x_2 \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x_2 & -x_1 \\ x_1 x_2 & x_1 \end{bmatrix}} R \longrightarrow 0. $$
Step 3: Compute the homology. Since the leftmost map has no kernel, we see $\Ext^0(R/I, R) = 0$. The kernel of the rightmost map is $R$; since we are computing homology we must quotient, and so

$$\Ext^2(R/I, R) \simeq R/(x_2, x_1) \simeq \mathbb{K}.$$ 

Note the only associated prime of $\Ext^2$ is $(x_1, x_2)$. Finally, we compute

$$\Ext^1(R/I, R) = \ker \left( [x_2, -x_1] \right) / \operatorname{im} \left( \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_1 x_2 \end{bmatrix} \right) = \left[ x_1 \right] / \left[ x_1^2 \right]$$

The last module has a single generator $e_1$, and the single relation is $x_1 \cdot e_1$, so we conclude that

$$\Ext^1(R/I, R) \simeq R/(x_1).$$

In particular, the only associated prime of $\Ext^1(R/I, R)$ is $(x_1)$, illustrating Theorem 4.1. For another example of this, see Chapter 9, Example 2.8.

Our next example brings the fine grading into the picture.

**Example 4.3.** Consider a $\mathbb{Z}^3$-filtration $\Delta$ on vertex set $\{a, b, c, d\}$ below, with all vertices appearing in degree 000, and

```
with edges in degrees
ab = 001  ac = 011  ad = 001  bc = 010  bd = 100  cd = 010
```

and triangles in degrees

```
abc = 011  abd = 101  acd = 011  bcd = 110.
```

In degree $(111)$ the triangles bound a hollow tetrahedron, so $H_2(\Delta_{111}) \neq 0$. To
see this algebraically, with respect to the ordered bases above we have

\[ d_2(\Delta) = \begin{bmatrix} x_2 & x_1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -x_1 & -x_2 & 0 \\ x_3 & 0 & 0 & x_1 \\ 0 & x_3 & 0 & -x_2 \\ 0 & 0 & x_3 & x_1 \end{bmatrix} \]

and

\[ d_1(\Delta) = \begin{bmatrix} -x_3 & -x_2 & -x_3 & 0 & 0 & 0 \\ x_3 & 0 & 0 & -x_2 & -x_1 & 0 \\ 0 & x_2 & x_3 & 0 & x_2 & -x_2 \\ 0 & 0 & x_3 & 0 & x_1 & x_2 \end{bmatrix} \]

Exercise 4.4. Show that \( H_2(\Delta) \cong \mathbb{R} \langle 1, 1, 1 \rangle \) and \( H_1(\Delta) = 0 \). \( \diamond \)

The second column of \( d_1 \) is a combination of the first and fourth columns so can be dropped. A minimal presentation matrix for \( H_0(\Delta) \) is given by

\[ \phi = \begin{bmatrix} 0 & 0 & 0 & -x_3 & -x_3 \\ -x_1 & -x_2 & 0 & x_3 & 0 \\ 0 & x_2 & -x_2 & 0 & 0 \\ x_1 & 0 & x_2 & 0 & x_3 \end{bmatrix} \]

A minimal free resolution for \( H_0(\Delta) \) is given by

\[ 0 \rightarrow R^1 \rightarrow R^3 \rightarrow R^5 \phi \rightarrow R^4 \rightarrow H_0(\Delta) \rightarrow 0. \]

Exercise 4.5. Use the free resolution above to prove

\[ \text{Ext}^3(H_0(\Delta), R) \cong \mathbb{K} \]
\[ \text{Ext}^2(H_0(\Delta), R) \cong 0 \]
\[ \text{Ext}^1(H_0(\Delta), R) \cong \text{coker} \left( \begin{bmatrix} x_2 & 0 \\ 0 & x_3 \end{bmatrix} \right) \]
\[ \text{Hom}(H_0(\Delta), R) \cong \text{coker} \left( \begin{bmatrix} x_2 & 0 \\ 0 & x_3 \end{bmatrix} \right) \]

To do this, we transpose the differentials in the free resolution for \( H_0(\Delta) \) and compute homology. For example, to see that \( \text{Ext}^3(H_0(\Delta), R) \cong \mathbb{K} \), just note that this module is the cokernel of transpose of the last map above

\[ \text{Ext}^3(H_0(\Delta), R) = R^1/\text{im}(\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}) \]

which is clearly \( \mathbb{K} \). \( \diamond \)
It will be useful to display the $R$-modules $\Ext^i(H_j(\Delta), R)$ in an $(i + 1) \times (j + 1)$ array as below

| $\Ext^3(H_0(\Delta), R)$ | $\Ext^3(H_1(\Delta), R)$ | $\Ext^3(H_2(\Delta), R)$ |
| $\Ext^2(H_0(\Delta), R)$ | $\Ext^2(H_1(\Delta), R)$ | $\Ext^2(H_2(\Delta), R)$ |
| $\Ext^1(H_0(\Delta), R)$ | $\Ext^1(H_1(\Delta), R)$ | $\Ext^1(H_2(\Delta), R)$ |
| $\Hom(H_0(\Delta), R)$ | $\Hom(H_1(\Delta), R)$ | $\Hom(H_2(\Delta), R)$ |

Since we have shown that $H_2(\Delta)$ is free rank one, for Example 4.3 this array takes the form

\[
\begin{array}{ccc}
\mathbb{K} & 0 & 0 \\
0 & 0 & 0 \\
\Ext^1(H_1(\Delta), R) & 0 & 0 \\
R^1 & 0 & R^1 \\
\end{array}
\]

In Chapter 9, we’ll see that the different spots in this diagram are connected by a spectral sequence: there is a map ($d_3$ for the experts) $\Hom_R(H_2(\Delta), R) \rightarrow \mathbb{K}$.
Chapter 9

Derived Functors and Spectral Sequences

This chapter gives a short, intense introduction to derived functors and spectral sequences, two powerful but sometimes intimidating topics. It is the most technical chapter of the book. It is aimed for the reader who is a non-specialist, so contains basic definitions and theorems (sometimes without proof) as well as illustrative examples. We apply these tools to multiparameter persistent homology.

Exercise 2.17 of Chapter 2 showed that tensoring a short exact sequence of $R$-modules

$$0 	o A_1 \xrightarrow{f} A_2 \to A_3 \to 0$$

with an $R$-module $M$ results in a sequence

$$A_1 \otimes_R M \xrightarrow{\bar{f}} A_2 \otimes_R M \to A_3 \otimes_R M \to 0$$

which is only exact on the right. What is the kernel of the induced map $\bar{f}$? Derived functors provide the answer; to set up the machinery we first need to define projective and injective objects, and resolutions. This chapter covers

- Injective and Projective Objects.
- Derived Functors.
- Spectral Sequences.
- Pas de deux: Spectral Sequences and Derived Functors.

Comprehensive treatments of the material here (and proofs) may be found in Eisenbud [47], Hartshorne [62], and Weibel [97]. Throughout this chapter, the ring $R$ is a commutative Noetherian ring with unit over a field $\mathbb{K}$, and the sheaf $\mathcal{O}_X$ is the sheaf of regular functions on a variety $X$. 
1. Injective and Projective Objects, Resolutions

1.1. Projective and Injective Objects.

**Definition 1.1.** A module $P$ is **projective** if it possesses a universal lifting property: For any $R$–modules $G$ and $H$, given a homomorphism $P \xrightarrow{\alpha} H$ and surjection $G \xrightarrow{\beta} H$, there exists a homomorphism $\theta$ making the diagram below commute:

\[\begin{array}{ccc}
P & \xrightarrow{\theta} & G \\
| & & \downarrow{\beta} \\
H & \xrightarrow{\alpha} & 0
\end{array}\]

Recall that a finitely-generated $R$-module $M$ is **free** if $M$ is isomorphic to a direct sum of copies of the $R$–module $R$. Free modules are projective.

**Definition 1.2.** A module $I$ is **injective** if given a homomorphism $H \xrightarrow{\alpha} I$ and injection $H \xrightarrow{\beta} G$, there exists a homomorphism $\theta$ making the diagram below commute:

\[\begin{array}{ccc}
I & \xrightarrow{\theta} & G \\
| & & \uparrow{\alpha} \\
H & \xleftarrow{\beta} & 0
\end{array}\]

Projective and Injective modules will come to the forefront in the next section, which describes derived functors.

1.2. Resolutions. Given an $R$–module $M$, there exists a projective module surjecting onto $M$; for example, take a free module with a generator for each element of $M$. This yields an exact sequence:

\[P_0 \xrightarrow{d_0} M \longrightarrow 0.\]

The map $d_0$ has a kernel, so the process can be iterated, producing an exact sequence (possibly infinite) of free modules, terminating in $M$.

**Definition 1.3.** A **projective resolution** for an $R$–module $M$ is an exact sequence of projective modules

\[\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0, \text{ with } \ker(d_1) = M.\]

Notice there is no uniqueness property; for example we could set $P'_2 = P_2 \oplus R$ and $P'_1 = P_1 \oplus R$, and define a map $P'_2 \to P'_1$ which is the identity on the $R$–summands, and the original map on the $P_i$ summands. In the category of $R$–modules the construction above shows that projective resolutions always exist. Surprisingly this is not the case for sheaves of $\mathcal{O}_X$-modules. In fact ([62], Exercise III.6.2), for $X = \mathbb{P}^1$ there is no projective object surjecting onto $\mathcal{O}_X$. 
Definition 1.4. An injective resolution for an $R$–module $M$ is an exact sequence of injective modules

$$I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \cdots \text{ with } \ker(d_0) = M.$$ 

It is not obvious that injective resolutions exist, but it can be shown (e.g. [47], [62], [97]) that in both the category of $R$–modules and in the category of sheaves of $\mathcal{O}_X$–modules, every object does have an injective resolution.

Exercise 1.5. Prove that in the category of $\mathbb{Z}$-modules (Abelian groups) that every object has an injective resolution. An injective $\mathbb{Z}$-module is called divisible; if you get stuck, see §IV.3 of [65].

Exercise 1.6. Prove that in a category in which every object includes in an injective object, injective resolutions of complexes always exist.

2. Derived Functors

In this section we describe the construction of derived functors, focusing on $\text{Ext}^i$ (in the category of $R$-modules) and $H^i$ (in the category of sheaves of $\mathcal{O}_X$–modules). For brevity we call these two categories “our categories”. Working in our categories keeps things concrete and lets us avoid introducing too much terminology, while highlighting the most salient features of the constructions, most of which apply in much more general contexts. For proofs and a detailed discussion, see [47] or [97]. We quickly review the definitions from Chapter 7:

2.1. Categories and Functors. Recall that a category is a class of objects, along with morphisms between the objects, satisfying certain properties: composition of morphisms is associative, and identity morphisms exist.

Definition 2.1. Suppose $\mathcal{B}$ and $\mathcal{C}$ are categories. A functor $F$ is a function from $\mathcal{B}$ to $\mathcal{C}$, taking objects to objects and morphisms to morphisms, preserving identity morphisms and compositions. If

$$B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3$$

is a sequence of objects and morphisms in $\mathcal{B}$, then

- $F$ is covariant if applying $F$ yields a sequence of objects and morphisms in $\mathcal{C}$ of the form:

$$F(B_1) \xrightarrow{F(b_1)} F(B_2) \xrightarrow{F(b_2)} F(B_3).$$

- $F$ is contravariant if applying $F$ yields a sequence of objects and morphisms in $\mathcal{C}$ of the form:

$$F(B_3) \xrightarrow{F(b_2)} F(B_2) \xrightarrow{F(b_1)} F(B_1).$$
A functor is **additive** if it preserves addition of homomorphisms; this property will be necessary in the construction of derived functors.

**Example 2.2.** The global sections functor of Chapter 4 is covariant: given a sequence of $\mathcal{O}_X$–modules

\[ \mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3, \]

taking global sections yields a sequence

\[ \Gamma(\mathcal{M}_1) \to \Gamma(\mathcal{M}_2) \to \Gamma(\mathcal{M}_3). \]

\[ \Diamond \]

**Definition 2.3.** Let $F$ be a functor from $\mathcal{B}$ to $\mathcal{C}$, with $\mathcal{B}$ and $\mathcal{C}$ categories of modules over a ring. Let

\[ 0 \longrightarrow B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3 \longrightarrow 0 \]

be a short exact sequence. $F$ is **left–exact** if either

(a) $F$ is covariant, and the sequence

\[ 0 \longrightarrow F(B_1) \xrightarrow{F(b_1)} F(B_2) \xrightarrow{F(b_2)} F(B_3) \]

is exact, or

(b) $F$ is contravariant, and the sequence

\[ 0 \longrightarrow F(B_3) \xrightarrow{F(b_2)} F(B_2) \xrightarrow{F(b_1)} F(B_1) \]

is exact.

A similar definition applies for right exactness; a functor $F$ is said to be exact if it is both left and right exact, which means that $F$ preserves exact sequences.

**2.2. Constructing Derived Functors.** The construction of derived functors is motivated by the following question: if $F$ is a left exact, contravariant functor and

\[ 0 \longrightarrow B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3 \longrightarrow 0 \]

is a short exact sequence, then what is the cokernel of $F(b_1)$? This object will appear on the right of $F(B_1)$ in (b) above. A left exact functor has **right derived functors**. Similarly, a right exact functor has **left derived functors**.

**Definition 2.4.** Let $\mathcal{B}$ be the category of modules over a ring, and let $F$ be a left exact, contravariant, additive functor from $\mathcal{B}$ to itself. If $M \in \mathcal{B}$, then there exists a projective resolution $P_\bullet$ for $M$.

\[ \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \]

Applying $F$ to $P_\bullet$ yields a complex:

\[ 0 \longrightarrow F(P_0) \xrightarrow{F(d_1)} F(P_1) \xrightarrow{F(d_2)} F(P_2) \longrightarrow \cdots \]
The **right derived functors** $R^i F(M)$ are defined as
\[ R^i F(M) = H^i(F(P_\bullet)). \]

**Theorem 2.5.** $R^i F(M)$ is independent of the choice of projective resolution, and has the following properties:

- $R^0 F(M) = F(M)$.
- If $M$ is projective then $R^i F(M) = 0$ if $i > 0$.
- A short exact sequence
  \[ B_\bullet : 0 \rightarrow B_1 \overset{b_1}{\rightarrow} B_2 \overset{b_2}{\rightarrow} B_3 \rightarrow 0 \]
  gives rise to a long exact sequence
  \[
  \begin{array}{ccc}
  R^{j-1} F(B_3) & \overset{R^{j-1}(F(b_2))}{\longrightarrow} & R^{j-1} F(B_2) \overset{R^{j-1}(F(b_1))}{\longrightarrow} R^{j-1} F(B_1) \\
  \downarrow{\delta_{j-1}} & & \downarrow{\delta_j} \\
  R^j F(B_3) & \overset{R^j(F(b_2))}{\longrightarrow} & R^j F(B_2) \overset{R^j(F(b_1))}{\longrightarrow} R^j F(B_1) \\
  \downarrow{\delta_{j+1}} & & \downarrow{\delta_{j+1}} \\
  R^{j+1} F(B_3) & \overset{R^{j+1}(F(b_2))}{\longrightarrow} & R^{j+1} F(B_2) \overset{R^{j+1}(F(b_1))}{\longrightarrow} R^{j+1} F(B_1)
  \end{array}
  \]
  of derived functors, where the connecting maps are natural: given another short exact sequence $C_\bullet$ and map from $B_\bullet$ to $C_\bullet$, the obvious diagram involving the $R^i F$ commutes.

The proof is almost exactly the same as the proof of Theorem 1.6 in Chapter 6; see Proposition A3.17 of [47]. There are four possible combinations of variance and exactness; the type of resolution used to compute the derived functors of $F$ is given below:

<table>
<thead>
<tr>
<th>$F$</th>
<th>covariant</th>
<th>contravariant</th>
</tr>
</thead>
<tbody>
<tr>
<td>left exact</td>
<td>injective</td>
<td>projective</td>
</tr>
<tr>
<td>right exact</td>
<td>projective</td>
<td>injective</td>
</tr>
</tbody>
</table>

**Exercise 2.6.** Prove that the derived functors do not depend on choice of resolution. The key to this is to construct a homotopy between resolutions, combined with the use of Theorem 1.6 of Chapter 6.

In the next sections, we study some common derived functors.
2.3. Ext. Let $R$ be a ring, and suppose

$$B_\bullet : 0 \to B_1 \xrightarrow{b_1} B_2 \xrightarrow{b_2} B_3 \to 0$$

is a short exact sequence of $R$–modules, with $N$ some fixed $R$–module. Applying $\text{Hom}_R(\bullet, N)$ to $B_\bullet$ yields an exact sequence:

$$0 \to \text{Hom}_R(B_3, N) \xrightarrow{c_1} \text{Hom}_R(B_2, N) \xrightarrow{c_2} \text{Hom}_R(B_1, N),$$

with $c_1(\phi) \mapsto \phi \circ b_2$ and $c_2(\theta) \mapsto \theta \circ b_1$; $\text{Hom}_R(\bullet, N)$ is left exact and contravariant.

**Definition 2.7.** $\text{Ext}^i_R(\bullet, N)$ is the $i^{th}$ right derived functor of $\text{Hom}_R(\bullet, N)$

Given $R$–modules $M$ and $N$, to compute $\text{Ext}^i_R(M, N)$, we must find a projective resolution for $M$

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0,$$

and compute the homology of the complex

$$0 \to \text{Hom}(P_0, N) \to \text{Hom}(P_1, N) \to \text{Hom}(P_2, N) \to \cdots$$

**Example 2.8.** Let $R = \mathbb{C}[x, y, z]$, $M = R/\langle xy, xz, yz \rangle$, and suppose $N \simeq R^1$. Applying $\text{Hom}_R(\bullet, R)$ to the projective (indeed, free) resolution of $M$

\[
\begin{pmatrix}
-x \\
y \\
z
\end{pmatrix}
\begin{pmatrix}
0 \\
x \\
y
\end{pmatrix}
\begin{pmatrix}
0 \\
-x \\
z
\end{pmatrix}
\begin{pmatrix}
0 \\
-x \\
z
\end{pmatrix}
\]

simply means dualizing the modules and transposing the differentials, so $\text{Ext}^i(R/I, R)$ is:

$$H_i \begin{pmatrix}
x y \\
x z \\
y z
\end{pmatrix} \to R(3)^2 \xrightarrow{\begin{pmatrix}
-z & y & 0 \\
-x & 0 & 0
\end{pmatrix}} R(2)^2 \to 0$$

Thus, $\text{Ext}^2(R/I, R)$ is the cokernel of the last map, and it is easy to check that $\text{Ext}^0(R/I, R) = \text{Ext}^1(R/I, R) = 0$. $\diamond$

For a fixed $R$–module $M$, applying $\text{Hom}_R(M, \bullet)$ to $B_\bullet$ yields an exact sequence:

$$0 \to \text{Hom}_R(M, B_1) \xrightarrow{c_1} \text{Hom}_R(M, B_2) \xrightarrow{c_2} \text{Hom}_R(M, B_3),$$

with $c_1(\phi) \mapsto b_1 \circ \phi$ and $c_2(\theta) \mapsto b_2 \circ \theta$; $\text{Hom}_R(\cdot, M)$ is left exact and covariant. Thus, to compute the derived functors of $\text{Hom}_R(\cdot, M)$, on a module $N$, we must find an injective resolution of $N$:

$$I^0 \to I^1 \to I^2 \to \cdots$$
2. Derived Functors

then compute

\[
H_i \left[ 0 \longrightarrow \text{Hom}(I^0, M) \longrightarrow \text{Hom}(I^1, M) \longrightarrow \text{Hom}(I^2, M) \longrightarrow \cdots \right]
\]

Using spectral sequences (next section), it is possible to show that \( \text{Ext}_i^i(M, N) \) can be regarded as the \( i \)th derived functor of either \( \text{Hom}_R(\bullet, N) \) or \( \text{Hom}_R(M, \bullet) \).

2.4. The global sections functor. Let \( X \) be a variety, and suppose \( \mathcal{B} \) is a coherent \( \mathcal{O}_X \)-module. The global sections functor \( \Gamma \) is left exact and covariant by Theorem 3.7 of Chapter 6. Hence, to compute \( R^i \Gamma(\mathcal{B}) \), we take an injective resolution of \( \mathcal{B} \):

\[
\mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots
\]

then compute

\[
H^i \left[ 0 \longrightarrow \Gamma(\mathcal{I}^0) \longrightarrow \Gamma(\mathcal{I}^1) \longrightarrow \Gamma(\mathcal{I}^2) \longrightarrow \cdots \right]
\]

In Example 2.8 we wrote down an explicit free resolution and computed the \( \text{Ext} \)-modules. Unfortunately, the general construction for injective resolutions produces very complicated objects. For example, if \( R = \mathbb{K}[x_1, \ldots, x_n] \), then the smallest injective \( R \)-module in which \( \mathbb{K} \) can be included is infinitely generated.

It is not obvious that there is a relation between the Čech cohomology which appeared in Chapter 6 and the derived functors of \( \Gamma \) defined above. Later in this chapter, we’ll see that there is a map

\[
\tilde{H}^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F}),
\]

and use spectral sequences to show that with certain conditions on \( \mathcal{U} \) this is an isomorphism. The upshot is that many key facts about cohomology such as Theorem 3.7 of Chapter 6 follow naturally from the derived functor machinery.

2.5. Acyclic objects. The last concept we need in order to work with derived functors is the notion of an acyclic object.

**Definition 2.9.** Let \( F \) be a left–exact, covariant functor. An object \( A \) is acyclic for \( F \) if \( R^i F(A) = 0 \) for all \( i > 0 \). An acyclic resolution of \( M \) is an exact sequence

\[
A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots
\]

where the \( A^i \) are acyclic, and \( M = \ker(d^0) \).

The reason acyclic objects are important is that a resolution of acyclic objects is good enough to compute higher derived functors; in other words we have an alternative to using resolutions by projective or injective objects.
Theorem 2.10. Let \( \mathcal{M} \) be a coherent \( \mathcal{O}_X \)-module, and
\[
\mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \mathcal{A}^2 \rightarrow \cdots
\]
a \( \Gamma \)-acyclic resolution of \( \mathcal{M} \). Then
\[
R^i \Gamma(\mathcal{M}) = H^i \left[ 0 \rightarrow \Gamma(\mathcal{A}^0) \rightarrow \Gamma(\mathcal{A}^1) \rightarrow \Gamma(\mathcal{A}^2) \rightarrow \cdots \right]
\]
Proof. First, break the resolution into short exact sequences:
\[
0 \rightarrow \mathcal{M} \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \mathcal{M}^0 \rightarrow 0
\]
Since the \( \mathcal{A}^i \) are acyclic for \( \Gamma \), applying \( \Gamma \) to the short exact sequence
\[
0 \rightarrow \mathcal{M} \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \mathcal{M}^0 \rightarrow 0
\]
yields an exact sequence
\[
0 \rightarrow \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{A}^0) \rightarrow \Gamma(\mathcal{A}^1) \rightarrow R^1 \Gamma(\mathcal{M}) \rightarrow 0
\]
Now apply the snake lemma to the middle two columns of the (exact, commutative) diagram below.
\[
0 \rightarrow \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{A}^0) \rightarrow \Gamma(\mathcal{A}^1) \rightarrow \Gamma(\mathcal{A}^2) \rightarrow \cdots
\]
This yields a right exact sequence
\[
0 \rightarrow R^1 \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{A}^1)/\Gamma(\mathcal{A}^0) \rightarrow \Gamma(\mathcal{M}^1) \simeq \Gamma(\mathcal{A}^1)/\ker(d^1),
\]
where \( \Gamma(\mathscr{A}^1) \xrightarrow{d^1} \Gamma(\mathscr{A}^2) \). Hence,

\[
R^1 \Gamma(\mathcal{M}) = H^1 \left[ \begin{array}{c}
0 \\
\Gamma(\mathscr{A}^0) \\
\Gamma(\mathscr{A}^1) \\
\Gamma(\mathscr{A}^2) \\
\vdots
\end{array} \right]
\]

In the next exercise you’ll show that iterating this process yields the theorem. □

**Exercise 2.11.** Complete the proof of Theorem 2.10 by replacing the sequence

\[
0 \to \mathcal{M} \to \mathscr{A}^0 \to \mathcal{M}^0 \to 0
\]

with

\[
0 \to \mathcal{M}^{i-1} \to \mathscr{A}^i \to \mathcal{M}^i \to 0
\]

\[\diamondsuit\]

### 3. Spectral Sequences

Spectral sequences are a fundamental tool in algebra and topology; when first encountered, they can seem quite confusing. In this brief overview, we describe a specific type of spectral sequence, state the main theorem, and illustrate the use of spectral sequences with several examples.

#### 3.1. Total complex of double complex.

**Definition 3.1.** A *first quadrant double complex* is a commuting diagram, where each row and each column is a complex:

\[
\cdots \xrightarrow{d_{03}} P_{02} \xleftarrow{\delta_{12}} P_{12} \xleftarrow{\delta_{22}} P_{22} \xleftarrow{\delta_{32}} \cdots
\]

\[
\cdots \xrightarrow{d_{13}} P_{01} \xleftarrow{\delta_{11}} P_{11} \xleftarrow{\delta_{21}} P_{21} \xleftarrow{\delta_{31}} \cdots
\]

\[
\cdots \xrightarrow{d_{23}} P_{00} \xleftarrow{\delta_{10}} P_{10} \xleftarrow{\delta_{20}} P_{20} \xleftarrow{\delta_{30}} \cdots
\]

For each antidiagonal, define a module

\[
P_m = \bigoplus_{i+j=m} P_{ij}.
\]

We may define maps

\[
P_m \xrightarrow{D_m} P_{m-1}
\]
via

\[ D_m(c_{ij}) = d_{ij}(c_{ij}) + (-1)^m \delta_{ij}(c_{ij}). \]

Thus, \( D_{m-1}D_m(a) = dd(a) + \delta\delta(a) = (d\delta(a) - \delta d(a)) \). The fact that each row and each column are complexes implies that \( \delta\delta(a) = 0 \) and \( dd(a) = 0 \). The commutativity of the diagram implies that \( d\delta(a) = \delta d(a) \), and so \( D^2 = 0 \).

**Definition 3.2.** The total complex \( \text{Tot}(P) \) associated to a double complex \( P_{ij} \) is the complex \( (P \ldots, D \ldots) \) defined above.

**Definition 3.3.** A filtration of a module \( M \) is a chain of submodules

\[ 0 \subseteq M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M \]

A filtration has an associated graded object \( \text{gr}(M) = \oplus M_i / M_{i+1} \). The main theorem concerning the spectral sequence of a double complex describes two different filtrations of the homology of the associated total complex. To describe these filtrations, we need to follow two different paths through the double complex.

### 3.2. The vertical filtration

For a double complex as above, we first compute homology with respect to the vertical differentials, yielding the following diagram:

\[
\begin{array}{c}
\ker(d_{02})/\text{im}(d_{03}) \xleftarrow{\delta_{12}} \ker(d_{12})/\text{im}(d_{13}) \xleftarrow{\delta_{22}} \ker(d_{22})/\text{im}(d_{23}) \xleftarrow{\delta_{32}} \\
\ker(d_{01})/\text{im}(d_{02}) \xleftarrow{\delta_{11}} \ker(d_{11})/\text{im}(d_{12}) \xleftarrow{\delta_{21}} \ker(d_{21})/\text{im}(d_{22}) \xleftarrow{\delta_{31}} \\
\end{array}
\]

\[
P_{00}/\text{im}(d_{01}) \xleftarrow{\delta_{10}} P_{10}/\text{im}(d_{11}) \xleftarrow{\delta_{20}} P_{20}/\text{im}(d_{21})
\]

These objects are renamed as follows:

\[
\begin{array}{c}
\text{vert}E^2_{02} \xleftarrow{\delta_{12}} \text{vert}E^2_{12} \xleftarrow{\delta_{22}} \text{vert}E^2_{22} \xleftarrow{\delta_{32}} \\
\text{vert}E^1_{01} \xleftarrow{\delta_{11}} \text{vert}E^1_{11} \xleftarrow{\delta_{21}} \text{vert}E^1_{21} \xleftarrow{\delta_{31}} \\
\text{vert}E^1_{00} \xleftarrow{\delta_{10}} \text{vert}E^1_{10} \xleftarrow{\delta_{20}} \text{vert}E^1_{20} \xleftarrow{\delta_{30}}
\end{array}
\]

The vertical arrows disappeared after computing homology with respect to \( d \), and the horizontal arrows reflect the induced maps on homology from the original diagram. Now, compute the homology of the diagram above, with respect to the horizontal maps. For example, the object \( \text{vert}E^2_{11} \) represents

\[
\ker(\text{vert}E^1_{11} \xrightarrow{\delta_{11}} \text{vert}E^1_{01})/\text{im}(\text{vert}E^1_{21} \xrightarrow{\delta_{21}} \text{vert}E^1_{11})
\]
The resulting modules may be displayed in a grid:

\[
\begin{array}{ccc}
\text{vert} E^2_{02} & \text{vert} E^2_{12} & \text{vert} E^2_{22} \\
\text{vert} E^2_{01} & \text{vert} E^2_{11} & \text{vert} E^2_{21} \\
\text{vert} E^2_{00} & \text{vert} E^2_{10} & \text{vert} E^2_{20}
\end{array}
\]

Although it appears at first that there are no maps between these objects, the crucial observation is that there is a map \(d^2_{i,j}\) from \(E^2_{i,j}\) to \(E^2_{i-2,j+1}\). This “knight’s move” is constructed just like the connecting map \(\delta\) appearing in the snake lemma. The diagram above (with differentials added) is thus:

\[
\begin{array}{ccc}
\text{vert} E^2_{02} & \text{vert} E^2_{12} & \text{vert} E^2_{22} \\
& d^2_{21} & \\
\text{vert} E^2_{01} & \text{vert} E^2_{11} & \text{vert} E^2_{21} \\
& d^2_{20} & \\
\text{vert} E^2_{00} & \text{vert} E^2_{10} & \text{vert} E^2_{20}
\end{array}
\]

So we may compute homology with respect to this differential. The homology at position \((i,j)\) is labeled, as one might expect, \(\text{vert} E^3_{i,j}\); it is now the case (but far from intuitive) that there is a differential \(d^3_{i,j}\) taking \(\text{vert} E^3_{i,j}\) to \(\text{vert} E^3_{i-3,j+2}\):

\[
\begin{array}{ccc}
\text{vert} E^3_{02} & \text{vert} E^3_{12} & \text{vert} E^3_{22} & \text{vert} E^3_{32} \\
& & \\
\text{vert} E^3_{01} & \text{vert} E^3_{11} & \text{vert} E^3_{21} & \text{vert} E^3_{31} \\
& d^3_{30} & \\
\text{vert} E^3_{00} & \text{vert} E^3_{10} & \text{vert} E^3_{20} & \text{vert} E^3_{30}
\end{array}
\]

The process continues, with \(d^r_{i,j}\) mapping \(\text{vert} E^r_{i,j}\) to \(\text{vert} E^r_{i-r,j+r-1}\). One thing that is obvious is that since the double complex lies in the first quadrant, eventually the differentials in and out at position \((i,j)\) must be zero, so that the module at position \((i,j)\) stabilizes; it is written \(\text{vert} E^\infty_{i,j}\). For example, it is easy to see that \(\text{vert} E^2_{10} = \text{vert} E^\infty_{10}\), while \(\text{vert} E^2_{20} \neq \text{vert} E^\infty_{20}\) but \(\text{vert} E^3_{20} = \text{vert} E^\infty_{20}\).
3.3. Main theorem. The main theorem is that the $E^\infty$ terms of a spectral sequence from a first quadrant double complex are related to the homology of the total complex.

Definition 3.4. If $\text{gr}(M)_m \simeq \bigoplus_{i+j=m} E^\infty_{ij}$, then we say that a spectral sequence of the filtered object $M$ converges, and write

$E^r \Rightarrow M$

Theorem 3.5. For the filtration of $H_m(\text{Tot})$ obtained by truncating columns of the double complex,

$\bigoplus_{i+j=m} \text{vert } E^\infty_{ij} \Rightarrow H_m(\text{Tot}).$

As with the long exact sequence of derived functors, the proof is not bad, but lengthy, so we refer to [97] or [47] for details. In §3.2, we first computed homology with respect to the vertical differential $d$. If instead we first compute homology with respect to the horizontal differential $\delta$, then the higher differentials are:

As before, for $r \gg 0$, the source and target are zero, so the homology at position $(i, j)$ stabilizes. The resulting value is denoted $\text{hor } E^\infty_{ij}$, and we have:

Theorem 3.6. For the filtration of $H_m(\text{Tot})$ obtained by truncating rows of the double complex,

$\bigoplus_{i+j=m} \text{hor } E^\infty_{ij} \Rightarrow H_m(\text{Tot}).$

For a first quadrant double complex, the two theorems above tell us that

$\bigoplus_{i+j=m} \text{hor } E^\infty_{ij} \Rightarrow H_m(\text{Tot})$ and $\bigoplus_{i+j=m} \text{vert } E^\infty_{ij} \Rightarrow H_m(\text{Tot}).$

Because the filtrations for the horizontal and vertical spectral sequence are different, it is often the case that for one of the spectral sequences the $E^\infty$ terms stabilize very early, perhaps even vanishing. So the main idea is to play off the two different filtrations against each other. This is illustrated in the next example.
Example 3.7. We prove Theorem 1.6 of Chapter 6 via spectral sequences. Let $0 \to C_2 \to C_1 \to C_0 \to 0$ be a short exact sequence of complexes:

$$
\begin{array}{ccc}
 & 0 & 0 & 0 \\
\downarrow & & & \\
C_2 : 0 & \leftarrow C_{02} & \leftarrow C_{12} & \leftarrow C_{22} \leftarrow & \\
\downarrow & & & \\
C_1 : 0 & \leftarrow C_{01} & \leftarrow C_{11} & \leftarrow C_{21} \leftarrow & \\
\downarrow & & & \\
C_0 : 0 & \leftarrow C_{00} & \leftarrow C_{10} & \leftarrow C_{20} \leftarrow & \\
\downarrow & & & \\
& 0 & 0 & 0 \\
\end{array}
$$

Since the columns are exact, it is immediate that for all $(i,j)$

$$\text{vert } E^1_{ij} = \text{vert } E^\infty_{ij} = 0$$

By Theorem 3.5, we conclude $H_m(\text{Tot}) = 0$ for all $m$. For the horizontal filtration $\text{hor } E^1_{ij} = H_i(C_j)$ if $j \in \{0, 1, 2\}$, and 0 otherwise. For $E^2$ we have

$$\text{hor } E^2_{ij} = \begin{cases} 
\ker(H_i(C_2) \to H_i(C_1)) & j = 2 \\
\ker(H_i(C_1) \to H_i(C_0))/\text{im}(H_i(C_2) \to H_i(C_1)) & j = 1 \\
\text{coker}(H_i(C_1) \to H_i(C_0)) & j = 0.
\end{cases}$$

The $d_2$ differential is zero for the middle row, and maps $\text{hor } E^2_{i,2} \to \text{hor } E^2_{i+1,0}$:

$$
\begin{array}{ccc}
\cdots & \text{hor } E^2_{i,2} & \text{hor } E^2_{i+1,2} & \cdots \\
\cdots & \text{hor } E^2_{i,1} & d_2 & \text{hor } E^2_{i+1,1} & \cdots \\
\cdots & \text{hor } E^2_{i,0} & \text{hor } E^2_{i+1,0} & \cdots \\
\end{array}
$$

So $\text{hor } E^2_{i,1} = \text{hor } E^\infty_{i,1}$, while

$$\text{hor } E^3_{i,2} = \text{hor } E^\infty_{i,2} = \ker(\text{hor } E^2_{i,2} \to \text{hor } E^2_{i+1,0})$$

and

$$\text{hor } E^3_{i,0} = \text{hor } E^\infty_{i,0} = \text{coker}(\text{hor } E^2_{i,2} \to \text{hor } E^2_{i+1,0})$$

By Theorem 3.6,

$$H_m(\text{Tot}) = \bigoplus_{i+j=m} \text{hor } E^\infty_{ij}$$
From the vertical spectral sequence, $H_m(\text{Tot}) = 0$, so all the terms $\text{hor} \ E_{ij}^\infty$ must vanish. Working backwards, we see this means

$$0 = \text{hor} \ E_{i,1}^2 = \ker(H_i(C_1) \to H_i(C_0))/\text{im}(H_i(C_2) \to H_i(C_1)),$$

hence $H_i(C_2) \to H_i(C_1) \to H_i(C_0)$ is exact, and

$$\ker(H_i(C_2) \to H_i(C_1)) \simeq \text{coker}(H_{i+1}(C_1) \to H_{i+1}(C_0))$$

which yields the long exact sequence in homology.

**Exercise 3.8.** Tensor product is right exact and covariant. Fix an $R$-module $N$. Prove that the $i$th left derived functor of $\bullet \otimes_R N$ is isomorphic to the $i$th left derived functor of $N \otimes_R \bullet$ as follows: Let

$$\cdots \to P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0$$

be a projective resolution for $M$ and

$$\cdots \to Q_2 \xrightarrow{q_2} Q_1 \xrightarrow{q_1} Q_0$$

be a projective resolution for $N$. Form the double complex

$$
\begin{array}{ccc}
\cdots & P_2 & P_1 & P_0 \\
\delta_{12} & \delta_{22} & \delta_{32} \\
& \delta_{11} & \delta_{21} & \delta_{31} \\
& & \delta_{10} & \delta_{20} & \delta_{30} \\
P_0 \otimes Q_2 & \leftarrow P_1 \otimes Q_2 & \leftarrow P_2 \otimes Q_2 & \cdots \\
\downarrow d_{02} & \downarrow d_{12} & \downarrow d_{22} \\
P_0 \otimes Q_1 & \leftarrow P_1 \otimes Q_1 & \leftarrow P_2 \otimes Q_1 & \cdots \\
\downarrow d_{01} & \downarrow d_{11} & \downarrow d_{21} \\
P_0 \otimes Q_0 & \leftarrow P_1 \otimes Q_0 & \leftarrow P_2 \otimes Q_0 & \cdots \\
\end{array}
$$

with differentials $P_i \otimes Q_j \xrightarrow{\delta_{ij}} P_{i-1} \otimes Q_j$ defined by $a \otimes b \mapsto p_i(a) \otimes b$, and $P_i \otimes Q_j \xrightarrow{d_{ij}} P_i \otimes Q_{j-1}$ defined by $a \otimes b \mapsto a \otimes q_j(b)$.

(a) Show that for the vertical filtration, the $E^1$ terms are

$$\text{vert} \ E_{ij}^1 = \begin{cases} 
P_i \otimes N & j = 0 \\
0 & j \neq 0. 
\end{cases}$$

Now explain why $\text{vert} \ E^2 = \text{vert} \ E^\infty$, and these terms are:

$$\text{vert} \ E_{ij}^2 = \begin{cases} 
H_i(P_\bullet \otimes N) = Tor_i(M,N) & j = 0 \\
0 & j \neq 0. 
\end{cases}$$
(b) Show that for the horizontal filtration
\[ \text{hor} E^2_{ij} = \begin{cases} H_j(M \otimes Q \bullet) = \text{Tor}_j(N, M) & i = 0 \\ 0 & i \neq 0. \end{cases} \]

(c) Put everything together to conclude that
\[ \text{Tor}_m(M, N) \simeq \bigoplus_{i+j=m} \text{vert} E^\infty_{ij} \simeq \text{gr}(H_m(\text{Tot})) \simeq \bigoplus_{i+j=m} \text{hor} E^\infty_{ij} \simeq \text{Tor}_m(N, M) \]

This fact is sometimes expressed by saying that Tor is a balanced functor.

Exercise 3.9. Prove that \( \text{Ext}^i(M, N) \) can be regarded as the \( i \)-th derived functor of either \( \text{Hom}_R(\bullet, N) \) or \( \text{Hom}_R(M, \bullet) \). The method is quite similar to the proof above, except for this one, you’ll need both projective and injective resolutions.

4. Pas de Deux: Spectral Sequences and Derived Functors

In this last section, we’ll see how useful the machinery of spectral sequences is in yielding theorems about derived functors. To do this, we first need to define resolutions of complexes. Note that sometimes our differentials on the double complex go “up and right” instead of “down and left”, so the higher differentials will change accordingly.

4.1. Resolution of a complex. Suppose
\[ A : 0 \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{A}^1 \longrightarrow \mathcal{A}^2 \longrightarrow \cdots \]
is a complex, either of \( R \)-modules or of sheaves of \( \mathcal{O}_X \)-modules. An injective resolution of \( A \) is a double complex as below:

\[ \begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array} \]

Let \( d^{jk} \) denote the horizontal differential at position \((j, k)\). The double complex is a resolution if it satisfies the following properties.
9. Derived Functors and Spectral Sequences

(a) The complex is exact.

(b) Each column \( J_i \) is an injective resolution of \( A_i \).

(c) \( \ker(d_{jk}) \) is an injective summand of \( J_{jk} \).

Condition (c) implies that \( \text{im}(d_{j,k}) \) is injective, yielding a “Hodge decomposition”:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{im}(d_{j-1,k}) & \longrightarrow & \ker(d_{j,k}) & \longrightarrow & H_{d}^{j,k} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{im}(d_{j-1,k}) & & & & & & & & \\
\end{array}
\]

It follows that we may decompose the sequence

\[
\begin{array}{ccccc}
\mathcal{J}_{j-1,k} & d_{j-1,k} & \mathcal{J}_{j,k} & d_{j,k} & \mathcal{J}_{j+1,k} \\
\end{array}
\]

as:

\[
\begin{array}{cccccccc}
\cdots \longrightarrow & \text{im}(d_{j-2,k}) & 0 & \longrightarrow & \text{im}(d_{j,k}) & 1 & \longrightarrow & \text{im}(d_{j,k}) & \cdots \longrightarrow \\
\oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots \longrightarrow & H_{j-1,k} & 0 & \longrightarrow & H_{j,k} & 0 & \longrightarrow & H_{j+1,k} & \cdots \longrightarrow \\
\oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots \longrightarrow & \text{im}(d_{j-1,k}) & 1 & \longrightarrow & \text{im}(d_{j-1,k}) & 0 & \longrightarrow & \text{im}(d_{j+1,k}) & \cdots \longrightarrow \\
\end{array}
\]

An inductive argument (Exercise 3.1) shows that in a category in which any object includes in an injective object, injective resolutions of complexes always exist.

4.2. Grothendieck spectral sequence. One of the most important spectral sequences is due to Grothendieck, and relates the higher derived functors of a pair of functors \( F, G \), and their composition \( FG \).

**Theorem 4.1.** Suppose that \( F \) is a left exact, covariant functor from \( \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) and \( G \) is a left exact, covariant functor from \( \mathcal{C}_2 \rightarrow \mathcal{C}_3 \). If \( A \in \mathcal{C}_1 \) has an \( F \)-acyclic resolution \( \mathcal{A}^\bullet \) such that \( F(\mathcal{A}^i) \) is \( G \)-acyclic then

\[
R^j G(R^i F(A)) \Rightarrow R^{i+j} G F(A)
\]

**Proof.** Take an injective resolution \( \mathcal{J}^{\bullet} \) for the complex

\[
\begin{array}{ccccccc}
0 & \longrightarrow & F(\mathcal{A}^0) & \longrightarrow & F(\mathcal{A}^1) & \longrightarrow & F(\mathcal{A}^2) & \longrightarrow & \cdots \\
\end{array}
\]
Apply $G$ to $\mathcal{F}^{\bullet\bullet}$. It follows from the construction above that a row of the double complex $G(\mathcal{F}^{\bullet\bullet})$ has the form:

$$
\begin{array}{c}
\cdots \to G(\text{im}(d^{i-2,k})) \xrightarrow{G^{(0)}} G(\text{im}(d^{i,k})) \xrightarrow{G^{(1)}} G(\text{im}(d^{i,k})) \cdots \\
\oplus \oplus \oplus \\
\cdots \to G(H^{i-1,k}) \xrightarrow{G^{(0)}} G(H^{i,k}) \xrightarrow{G^{(0)}} G(H^{i+1,k}) \cdots \\
\oplus \oplus \oplus \\
\cdots \to G(\text{im}(d^{i-1,k})) \xrightarrow{G^{(1)}} G(\text{im}(d^{i-1,k})) \xrightarrow{G^{(0)}} G(\text{im}(d^{i+1,k})) \cdots
\end{array}
$$

Hence,

$$\text{hor} E^1_{ij} = G(H^{i,j})$$

By construction, $H^{i,j}$ is the $j^{th}$ object in an injective resolution for the $i^{th}$ cohomology of $F(\mathcal{A}^{\bullet})$. Since $\mathcal{A}^{\bullet}$ was an $F$–acyclic resolution for $A$, the $i^{th}$ cohomology is exactly $R^i F(A)$, so that

$$\text{hor} E^2_{ij} = H^j \left[ 0 \to G(H^{i,0}) \to G(H^{i,1}) \to G(H^{i,2}) \to \cdots \right] = R^j G(R^i F(A))$$

Next, we turn to the vertical filtration. We have the double complex

$$
\begin{array}{c}
G(\mathcal{A}^{02}) \to G(\mathcal{A}^{12}) \to G(\mathcal{A}^{22}) \cdots \\
G(\mathcal{A}^{01}) \to G(\mathcal{A}^{11}) \to G(\mathcal{A}^{21}) \cdots \\
G(\mathcal{A}^{00}) \to G(\mathcal{A}^{10}) \to G(\mathcal{A}^{20}) \cdots
\end{array}
$$

Since $\mathcal{A}^{ij}$ is an injective resolution of $F(\mathcal{A}^i)$,

$$R^j G(F(\mathcal{A}^i)) = H^j \left[ 0 \to G(\mathcal{A}^{i,0}) \to G(\mathcal{A}^{i,1}) \to G(\mathcal{A}^{i,2}) \to \cdots \right].$$

Now, the assumption that the $F(\mathcal{A}^i)$ are $G$–acyclic forces $R^j G(F(\mathcal{A}^i))$ to vanish, for all $j > 0$! Hence, the cohomology of a column of the double complex above vanishes, except at position zero. In short

$$\text{vert} E^1_{ij} = \begin{cases} G(\mathcal{A}^i) & j = 0 \\ 0 & j \neq 0. \end{cases}$$
Thus,

\[ \text{vert} E_\infty^{ij} = \text{vert} E^2 = \begin{cases} R^{i+j}GF(A) & j = 0 \\ 0 & j \neq 0. \end{cases} \]

Applying Theorem 3.5 and Theorem 3.6 concludes the proof. \(\square\)

**Exercise 4.2.** Let \( Y \xrightarrow{f} X \) be a continuous map between topological spaces, with \( \mathcal{F} \) a sheaf on \( Y \). The pushforward is defined via:

\[ f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \]

(a) Show that pushforward \( f_* \) is left exact and covariant, so associated to \( \mathcal{F} \) are objects \( R^j f_*(\mathcal{F}) \).

(b) Use Theorem 4.1 to obtain the Leray spectral sequence:

\[ H^i(R^j f_*(\mathcal{F})) \Rightarrow H^{i+j}(\mathcal{F}) \]

What can you say when \( Y \) is the fiber of a vector bundle \( X \)? \(\diamondsuit\)

### 4.3. Comparing cohomology theories.

Our second application of spectral sequences will be to relate the higher derived functors of \( \Gamma \) to the Čech cohomology. If \( I_p \) denotes a \( p+1 \)-tuple \( \{i_0 < i_1 < \cdots < i_p\} \) and \( U_{I_p} = U_{i_0} \cap \cdots \cap U_{i_p} \), then applying the pushforward construction from Exercise 4.2 to the inclusion \( U_{I_p} \hookrightarrow X \) gives a sheaf theoretic version of the Čech complex.

\[ C^p(\mathcal{U}, \mathcal{F}) = \prod_{I_p} i_* (\mathcal{F}|_{U_{I_p}}). \]

**Exercise 4.3.** Show that \( C^* \) is a resolution of \( \mathcal{F} \), as follows. By working at the level of stalks, show that there is a morphism of complexes

\[ C^i(\mathcal{U}, \mathcal{F})_p \xrightarrow{k} C^{i-1}(\mathcal{U}, \mathcal{F})_p \]

such that \((d_{i-1} + k + kd_i)\) is the identity. Conclude by applying Theorem 1.8. Finally, show that if \( \mathcal{F} \) is injective, then so are the sheaves \( C^i(\mathcal{U}, \mathcal{F}) \). If you get stuck, see [62, III.4]. \(\diamondsuit\)

**Lemma 4.4.** For an open cover \( \mathcal{U} \), there is a map \( \tilde{H}^i(\mathcal{U}, \mathcal{F}) \to H^i(X, \mathcal{F}) \).

**Proof.** Take an injective resolution \( \mathcal{I}^* \) for \( \mathcal{F} \). By injectivity, we get

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{I}^* & \rightarrow & \mathcal{C}^0 & \rightarrow & \mathcal{C}^1 & \rightarrow & \cdots \\
& & & \uparrow & & & & & \\
& & & \mathcal{F}^0 & & & & & \\
\end{array}
\]

Iterating the construction gives a map of complexes \( \mathcal{C}^* \to \mathcal{I}^* \), which by Lemma 1.4 yields a corresponding map on cohomology. \(\square\)
Theorem 4.5. Let $\mathcal{U}$ be a Leray cover: an open cover such that for any $I_p$,
$$H^1(U_{I_p}, F) = 0,$$
for all $i \geq 1$.

Then
$$\tilde{H}^i(\mathcal{U}, F) = H^i(X, F).$$

**Proof.** Take an injective resolution $I^\bullet$ for $F$. Since $H^i(U_{I_p}, F) = 0$ for $i > 0$
$$0 \rightarrow F(U_{I_p}) \rightarrow I^0(U_{I_p}) \rightarrow I^1(U_{I_p}) \rightarrow \cdots$$
is exact. Then as in the construction of the sheaf-theoretic Čech complex, we obtain

a Čech complex built out of the direct product of these, which is by construction a resolution (depicted below) of the Čech complex for $F$. The bottom row is included for clarity, it is *not* part of the complex.

$$
\begin{array}{ccc}
\uparrow & & \\
\mathcal{C}^0(\mathcal{U}, I^2) & \rightarrow & \mathcal{C}^1(\mathcal{U}, I^2) & \rightarrow & \mathcal{C}^2(\mathcal{U}, I^2) & \rightarrow & \cdots \\
\uparrow & & & & & & \\
\mathcal{C}^0(\mathcal{U}, I^1) & \rightarrow & \mathcal{C}^1(\mathcal{U}, I^1) & \rightarrow & \mathcal{C}^2(\mathcal{U}, I^1) & \rightarrow & \cdots \\
\uparrow & & & & & & \\
\mathcal{C}^0(\mathcal{U}, I^0) & \rightarrow & \mathcal{C}^1(\mathcal{U}, I^0) & \rightarrow & \mathcal{C}^2(\mathcal{U}, I^0) & \rightarrow & \cdots \\
\uparrow & & & & & & \\
\mathcal{C}^0(\mathcal{U}, F) & \rightarrow & \mathcal{C}^1(\mathcal{U}, F) & \rightarrow & \mathcal{C}^2(\mathcal{U}, F) & \rightarrow & \cdots \\
\end{array}
$$

Applying $\Gamma$, since $H^i(U_{I_p}, F) = 0$ for $i > 0$,
$$\text{vert } E^1_{ij} = \begin{cases} 
\Gamma(\mathcal{C}^i(\mathcal{U}, F)) & j = 0 \\
0 & j \neq 0.
\end{cases}$$
Thus, $E^2 = E^\infty$, and since $\Gamma(\mathcal{C}^i(\mathcal{U}, F)) = C^i(\mathcal{U}, F)$
$$\text{vert } E^2_{ij} = \begin{cases} 
H^i(\mathcal{U}, F) & j = 0 \\
0 & j \neq 0.
\end{cases}$$

For the horizontal filtration, since the $\mathcal{C}^i(\mathcal{U}, I^j)$ are injective,
$$\text{hor } E^1_{ij} = \begin{cases} 
\Gamma(I^j) & i = 0 \\
0 & i \neq 0.
\end{cases}$$
and thus
$$\text{hor } E^2_{ij} = \begin{cases} 
H^j(\Gamma(I^j)) & i = 0 \\
0 & i \neq 0.
\end{cases}$$

This is the usual derived functor cohomology, and applying Theorem 3.5 and Theorem 3.6 concludes the proof. □
4.4. Cartan-Eilenberg Resolution. A Cartan-Eilenberg resolution of a complex is a resolution constructed in a fashion similar to the injective resolution of a complex appearing in §4.1. For a complex $C\bullet$, take free resolutions $F_{i+1}\bullet$ for $\text{im}(\partial_{i+1})$ and $G_i\bullet$ for $H_i(C)$. The short exact sequence

$$0 \to \text{im}(\partial_{i+1}) \to \ker(\partial_i) \to H_i(C) \to 0$$

combined with the resolutions above yields a resolution $G_i\bullet \oplus F_{i+1}\bullet$ for $\ker(\partial_i)$. Now use the resolutions for $\ker(\partial_i)$ and $\text{im}(\partial_i)$ and the short exact sequence

$$0 \to \ker(\partial_i) \to C_i \to \text{im}(\partial_i) \to 0$$

to obtain a resolution for $C_i$. The resulting double complex has terms

$$F_{i+1,j} \oplus G_{i,j} \oplus F_{i,j} \xrightarrow{d_{i,j}} F_{i,j} \oplus G_{i-1,j} \oplus F_{i-1,j}$$

with $d_{i,j}$ the identity on $F_{i,j}$ and zero on the other summands.

**Example 4.6.** Our final spectral sequence example comes from the three-parameter filtration of $S^2$ appearing in Example 4.3 in Chapter 8.

```
with edges in degrees
ab = 001  ac = 011  ad = 001  bc = 010  bd = 100  cd = 010
and triangles in degrees
abc = 011  abd = 101  acd = 011  bcd = 110.
```
Take a Cartan-Eilenberg resolution of the multiparameter persistent homology complex appearing in Definition 1.3 of Chapter 8. Now apply $\text{Hom}_R(\cdot, R)$, and compute the spectral sequences for the resulting double complex. The vertical filtration degenerates immediately, because

$$\begin{align*}
\text{vert}E^1_{ij} &= \text{Ext}^j(C_i, R) = \begin{cases} 
\text{Hom}_R(C_i, R) & j = 0 \\
0 & j \neq 0,
\end{cases}
\end{align*}$$

which follows since the $C_i$ are free modules, so $\text{Ext}^j(C_i, R)$ vanishes if $j \neq 0$.

Let $C^*_* \equiv \text{Complex of modules}$ $C^*_* = \text{Hom}_R(C_i, R)$, we have

$$(4.1) \quad \text{vert}E^2_{ij} = \text{vert}E^\infty_{ij} = \begin{cases} 
H_i(C^*_) & j = 0 \\
0 & j \neq 0,
\end{cases}$$

Transposing the differentials we found in Chapter 8 and computing homology we have

$$\begin{align*}
\text{vert}E^\infty_{00} &\simeq R^1 \\
\text{vert}E^\infty_{10} &\simeq R/\langle x_2 \rangle \oplus R/\langle x_3 \rangle \\
\text{vert}E^\infty_{20} &\simeq \ker([x_3, x_1, -x_2])
\end{align*}$$

For the horizontal filtration, the Cartan-Eilenberg construction yields

$$\text{vert}E^1_{ij} = G^*_{i,j}.$$  

Since the $G^*_{i,j}$ terms are a free resolution for $H_i$, dualizing and computing homology means that the resulting the $E^2$ terms are

$$(4.2) \quad \text{hor}E^2_{ij} = \text{Ext}^j(H_1(\Delta), R)).$$

In Exercise 4.4 and 4.5 of Chapter 8, we computed that for Example 4.3 the $E^2$ terms for the horizontal filtration are

$$\begin{array}{cccc}
\mathbb{K} & 0 & 0 \\
0 & 0 & 0 \\
\text{Ext}^1(H_0(\Delta), R) & 0 & 0 \\
R^1 & 0 & R^1
\end{array}$$

The bottom row follows from $\text{Hom}_R(H_0(\Delta), R) \simeq R^1 \simeq \text{Hom}_R(H_2(\Delta), R)$. There is a potentially nonzero $d_3$ differential

$$\text{Hom}_R(H_2(\Delta), R) \simeq R^1 \xrightarrow{d_3} \text{Ext}^3_R(H_0(\Delta), R) \simeq \mathbb{K}.$$ 

To analyze this map, we use that the horizontal and vertical filtrations both yield associated graded objects for the direct sum of the diagonals. And indeed,

$$\begin{align*}
\text{vert}E^\infty_{00} &\simeq R^1 \\
\text{vert}E^\infty_{10} &\simeq R/\langle x_2 \rangle \oplus R/\langle x_2 \rangle = \text{Ext}^1(H_0(\Delta), R) \simeq \text{hor}E^\infty_{01}
\end{align*}$$

Now we come to the interesting comparison. For the vertical filtration

$$\bigoplus_{i+j=2} \text{vert}E^\infty_{ij} \simeq \ker([x_3, x_1, -x_2])$$
But this must be the associated graded of a filtration for $H_2(Tot)$. On the other hand,

$$\bigoplus_{i+j=2} \text{hor} E_{ij}^\infty \simeq \ker(d_3)$$

So this means that we have a nontrivial $d_3$ differential mapping

$$R^1 \to \mathbb{K} = R^1 / \langle x_1, x_2, x_3 \rangle,$$

yielding

$$\ker(d_3) = \ker([x_3, x_1, -x_2])$$

By Theorem 4.1 the modules $\text{Ext}^j(H_i(\Delta), R)$ encode the codimension $j$ support locus of $H_i(\Delta)$. This means they provide a natural way to stratify MPH modules.

The Hilbert Syzygy Theorem shows that the resolution for a $\mathbb{K}[x_1, \ldots, x_n]$-module has length at most $n$, so

$$\text{Ext}^j(H_i(\Delta), R) = 0$$

when $j \geq n + 1$. Combining this with Equations 4.1 and 4.2 proves

**Theorem 4.7.** For a Cartan-Eilenberg resolution of the MPH complex $C_\bullet$ of Definition 1.3 of Chapter 8, we have

$$\bigoplus_{i+j=k} \text{hor} E_{ij}^\infty \Longrightarrow H_k(C^\bullet),$$

and

$$\text{hor} E_{ij}^m = \text{hor} E_{ij}^\infty \text{ for all } m \geq n + 1.$$ 

**Exercise 4.8.** Carry out this analysis for other examples. Notice for a filtration with $r$-parameters, $\text{Ext}^j(H_i(\Delta), R)$ vanishes for $j > r$, which means that for the horizontal filtration the differentials $d_j$ vanish for $j > r$. ◊
Bibliography


