Week 2a problem bank

1. Without using the distance formula, find an equation corresponding to the points (x, y, z) which are equidistant (i.e., have the same distance) from $(-2, 1, 4)$ and $(6, 3, -2)$. Simplify as much as possible.
2. Find the line in parametric form containing the point $(0, 1, 2)$ and which perpendicularly intersects the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.
3. A plane can be described as the set of points of the form $\langle x_0, y_0, z_0\rangle + s\langle a, b, c\rangle + t\langle d, e, f\rangle$ where s and t run over all possibilities and $\langle a, b, c\rangle$, $\langle d, e, f\rangle$ are non-parallel vectors in the plane. Rewrite the plane $\langle 0, -2, 3\rangle + s\langle 1, -1, 2\rangle + t\langle 3, 0, 1\rangle$ in the form $Ax + By + Cz = D$.
4. Find the line that *intersects* the line $\langle 1 + t, 2 - 3t, 2t\rangle$ perpendicularly and lies in the plane $x + y - 3z = -1$.
5. Find the point on the plane $x + 2y = 5 + 3z$ closest to the point $(4, 4, -7)$.
6. While holding an egg you travel along $\mathbf{r}(t) = (t^3 + t)\mathbf{i} + (6t^2 - 7)\mathbf{j} + (7 - t^3)\mathbf{k}$. At time $t = 1$ you let go of the egg which then continues on a straight line. Determine where the egg will hit the xy-plane.
7. Find the line in parametric form which is tangent to the curve $x = 2t^2 + t$, $y = 2t^2$, $z = t^3 - t$ at time $t = 2$.
8. Let $\mathbf{r}(t) = \langle 2\cos(\pi t), \sin(\pi t), t^3\rangle$. Find the plane perpendicular to this curve at time $t = 2$.
9. Find all times t so that the tangent line to the parametric curve $(t - 8, t^2 - 5t - 8, 48 - t^2)$ at time t passes through the origin.
10. Let $h(t) = \langle \sin(t) + 2t + 2, e^{-9t} - 4\cos(t), t^3 + 6t\rangle \times \langle 1 + \arctan(t), 7t^4 + t - 3, 2\cos(5t)\rangle$. Find $h'(0)$.

Week 2a additional practice

The following problems are drawn from a variety of sources, including previous exams and reviews. These are good practice to prepare for the upcoming exam. The more problems that one is *able* to answer, the better.

1. [F17-E1] Find the line which passes through the point $(2, 1, 4)$, is parallel to the plane $4x + 5z = 19 + 3y$, and intersects the z-axis.

2. Find the plane which bisects the following three line segments: between $(0, 2, -1)$ and $(4, 0, -5)$; between $(-2, 1, 4)$ and $(4, -1, 2)$; and between $(-1, 3, 1)$ and $(3, 1, 1)$.

3. Give a formula and name for the quadric surface on which the parametric curve $\big(e^t(\sin t + \cos t), e^t, e^{2t}\sin(2t)\big)$ lies.

4. Find all times t where the curve given by $\big(t^3 + e^t + 2, -t + \cos t + 1, 2t + e^t + 2\cos t\big)$ intersects the plane $x + 2y - z = 4$.

5. [S18-F] Find the plane that contains the line L given by $x = 3 + 2t$, $y = 1 - t$, $z = 2 + t$ and also bisects the line segment between the points $(2, 5, 3)$ and $(4, 9, -1)$. (To bisect the line segment means the plane will pass through the point midway between the endpoints of the segment.) [*Hint: The point of bisection is not on the line* L.]

6. [F16-F] (a) Find the point of intersection P of the lines

$$\begin{aligned} x &= 3 + t \\ y &= 4 + 3t \\ z &= 1 + 2t \end{aligned} \qquad\qquad \begin{aligned} x &= -2 + 2s \\ y &= 5 - 2s \\ z &= 1 - s \end{aligned}$$

(b) Find an equation for the plane containing the lines in (a).

7. [S16-F] (a) Find an equation for the plane that passes through the point $(1, 2, 3)$ and perpendicular to the line $x = 1 + 3t$, $y = 3 + 2t$, $z = 1 - t$.

(b) Find the distance from the point $(3, 2, 1)$ to the plane found from part (a)

8. [S15-F] Find a) an equation of the plane passing through the points $P(1, 0, 2)$, $Q(2, 1, 0)$, and $R(2, 2, 1)$ and b) the area of the triangle ΔPQR.

9. [F14-F] Find the distance from the point $P(1, 1, 0)$ to the plane containing the points $Q(2, 1, 0)$, $R(0, -1, 1)$, and $S(1, 0, -1)$.

10. [F13-F] The two curves

$$\mathbf{r}(t) = \big\langle t^2, 2-t, t^3+t-1\big\rangle \quad \text{and} \quad \mathbf{s}(t) = \big\langle t^4, 4-2t-t^2, t\big\rangle$$

both pass through the point $(1, 1, 1)$ at time $t = 1$. Find the *unique* plane which is tangent to both curves at the point $(1, 1, 1)$. (Hint: the normal vector of this plane must be perpendicular to the direction of motion of the curves at $t = 1$.)

11. [F12-F] (a) Find an equation for the plane through the points $A = (-2, 0, 2)$, $B = (2, 1, 0)$, and $C = (1, 2, 1)$.

(b) Find the point where this plane intersects the y-axis.

12. [S12-F] (a) Find a direction vector for the line of intersection of the planes $x + 2y + z = 0$ and $x + y + 1 = 0$.

(b) Find the equation of the plane containing the line $x = 2t$, $y = 1 - t$, $z = 3t$ and the vector found in (a).

13. [S12-F] When a curve intersects itself the angle of intersection is the angle between the two tangent vectors of the curve at the point of intersection. Find $\cos\theta$ where θ is the angle of intersection of the parametric curve $(t^2 - t, t^3 - 3t^2 + 2t)$ at the point $(0, 0)$ (corresponding to the intersection times $t = 0$ and $t = 1$).

14. Find the plane which contains the line $\langle 3, 1, 2\rangle + t\langle 1, -2, 4\rangle$ and the line $\langle 2, -3, 3\rangle + s\langle 2, -4, 8\rangle$.

15. For $t \geqslant 0$, find a *unit* vector, as a function of t, which points in the direction of acceleration given that position is given by
$\mathbf{r}(t) = \langle t\sin t + \cos t, t\cos t - \sin t, \frac{4\sqrt{2}}{15}t^{5/2}\rangle$.

16. Let ℓ_1 be the line segment joining $(3,3,2)$ with $(-1,5,0)$; let ℓ_2 be the line segment joining $(1,4,3)$ with $(-3,2,1)$; let ℓ_3 be the line segment joining $(4,7,6)$ with $(-2,-3,2)$. Find the *unique* plane which would cut these three line segments in half (i.e., passes through their midpoints).

    (This is a special case of what is known as the Ham Sandwich Theorem.)

17. Find the tangent line to
$r(t) = \left\langle e^{2t} - 3t, \sin(2t) - \cos(t), t^2 - 2t + 4\right\rangle$ at $t = 0$. Give your answer in parametric form.

18. The point $(2,1,3)$ is the midpoint of the points P and Q. If the point P lies on the line $\langle x,y,z\rangle = \langle 5,2,4\rangle + t\langle 2,-3,1\rangle$, then Q lies on some other line. Determine this other line and give it in *parametric form*.

19. Find the osculating plane to
$r(t) = \left\langle e^{2t} - 3t, \sin(2t) - \cos(t), t^2 - 2t + 4\right\rangle$ at $t = 0$. (Recall the osculating plane contains the point, the direction of motion, and the direction of acceleration.)

20. Find the line which passes through the point $(3,-2,4)$ and *does not* intersect either the plane $3x - y + 2z = 5$ or the plane $x + 3y - 5z = 2$. Give your answer in parametric form.

21. Tony Stark as Iron Man has found himself in an intense fight with Obadiah Stone who has turned on Stark and adapted the persona of Iron Monger. Currently Iron Man and Iron Monger are flying along the curves

$$\mathbf{r}(t) = \langle t^2 - 3, t^2 - 2t, 3t - 3\rangle \text{ and}$$
$$\mathbf{s}(t) = \langle 7 - 3t, 4t - 8, 5 - 3t + t^2\rangle$$

    respectively and they will collide at $t = 2$. To help prepare for the impact J.A.R.V.I.S. (the artificial intelligence program that helps assist the control of the suit) is calculating the angle of impact when the collision will occur. Unfortunately Iron Man's suit has taken some damage and so not all of the features are currently working (i.e., for some bizarre reason the arc-cosine function is not available), however J.A.R.V.I.S. is still able to perform other simple computations.

    Determine the value of $\cos\theta$ where θ will be the angle of impact (i.e., the angle between their respective directions of motion at the time of impact).

22. (a) Find the line in parametric form which is parallel to the plane $3x + z = 2y + 137$, passes through the point $(2,0,1)$ and perpendicularly intersects the line $x = 2 - t$, $y = 2t$, $z = 1 + t$.

    (b) For the line found in part (a), find the point(s) where it intersects the xy-plane and the point(s) where it intersects the yz-plane.

23. Find the equations of the two planes which are perpendicular to the plane $x + 2y - 3z = 7$ and also intersect the curve $(t^3, 2 - 3t, \frac{1}{2}t^2 - 2t + \frac{3}{2})$ perpendicularly.

24. Find the plane that contains the line

$$\frac{x-2}{3} = 2 - y = \frac{2z-1}{4}$$

    and is perpendicular to the plane $2x + 2y = z + \pi^3$.

    (Note this form of expressing a line is known as *symmetric* form. To convert to parametric form set each individual term to t and solve.)

25. You have recently made it onto the newest reality show called *The Ultimate Extreme Roller Coaster Happy Fun Hour*. You are currently in the second round and in this round there is a balloon filled with paint attached to the bottom of the roller coaster that you must detach by remote control so that it falls and hits a target.

    As you wait for your turn you watch the contestants in front of you and cannot help but notice that if the target corresponded to $(0,0,0)$ then the position of the roller coaster could be described by $\left(t - 8, t^2 - 5t - 8, 48 - t^2\right)$ for $0 \leqslant t \leqslant 6$ (the drop zone). Neglecting gravity and wind resistance (after all this is a "reality" show), at what time t should you release the balloon to hit a perfect bullseye?

    (Hint: to hit a perfect bullseye what must be true about the relationship between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$?)

26. The following two lines are intersecting:

$$\begin{aligned} x &= 1 - t \\ y &= 2 + t \\ z &= 3 - 2t \end{aligned} \qquad \begin{aligned} x &= t \\ y &= 3 - t \\ z &= 3 + t \end{aligned}$$

    Find the line which goes through the intersection point and is perpendicular to both given lines.

27. When two parametric curves intersect, their angle of intersection is the angle between their tangent vectors. Find $\cos\theta$, where θ is the angle of intersection of the curves $(e^{t-1}, t^2, \ln(t))$ and $(t^3, 2 - t, \sin(\pi t))$ at time $t = 1$.

28. [F18-E1] (a) Find an equation for the plane passing through the points $A(1,3,8)$, $B(-2,-4,7)$, and $C(5,-1,-4)$.

    (b) Find the area of the triangle with vertices A, B, and C.

29. You have recently made it onto the newest reality show called *The Ultimate Extreme Roller Coaster Happy Fun Hour*. You are currently in the third round and in this round you ride the roller coaster holding onto a balloon filled with paint. At a precise moment you must throw the balloon with a goal of hitting the target.

As you wait for your turn you watch the contestants in front of you and cannot help but notice that if the target corresponded to $(0,0,0)$ then the position of the roller coaster could be described by $\left(t-8, t^2-5t-8, 48-t^2\right)$ for $0 \leqslant t \leqslant 6$ (the drop zone). Neglecting gravity and wind resistance (after all this is a "reality" show), how should you throw the balloon (as a vector) at time $t=2$ so that it will hit the bullseye exactly two seconds after you throw it?

(Hint: to hit a perfect bullseye what must be true about the relationship between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$?)

30. [F18-E1] Find the osculating plane to the curve $\mathbf{r}(t)$ at time $t=0$. (Recall that the osculating plane contains the point on the curve, the direction of velocity, and the direction of acceleration.)

$$\mathbf{r}(t) = \left(\tan(t)+2+t^3\right)\mathbf{i} + \left(1+t^2\right)\mathbf{j} + \left(\sin(t)+\cos(t)\right)\mathbf{k}$$

31. [F18-E1] The position function of a spaceship is given by:

$$\mathbf{r}(t) = \left\langle 3+t, t^2-7, t^2+2t\right\rangle$$

At time $t=0$ (in seconds), the ship's captain receives orders to turn off the ship's engines **within the next six seconds** and then have the ship coast to a space station located at the point $(8,9,26)$.

Determine the time t when the engines should be turned off, *and* give the location of the ship at that moment in time.

32. [S19-E1] A line passes through the origin and through the midpoint between $A=(3,1,6)$ and $B=(1,5,2)$. Find $\cos\theta$ where θ is the angle between the line and the line segment connecting A and B.

33. Find the line which is contained in the plane $x+y-3z=7$ and perpendicularly intersects the line $\langle 3+5t, 2-4t, -2+t\rangle$.

34. [S19-F] Find *both* lines which intersect the z-axis *and* are tangent to the curve $\mathbf{r}(t) = \left\langle t^2-3t, t+1, t^3+5t\right\rangle$. You can give the answer(s) in either vector form or parametric form; simplify your answers. (Hint: points on the z-axis are of the form $(0,0,z)$.)

35. [F17-E1] Find the plane which contains the points $(1,3,2)$ and $(2,1,4)$ and also perpendicularly intersects the plane $3x+y=5z+19$.

36. [F17-E1] Find the osculating plane to the curve $\mathbf{r}(t) = \langle \frac{3}{2}t^2-4t+5, e^{3t}+2\cos(2t), \sin(t)\rangle$ at time $t=0$. (Recall that the osculating plane contains both the direction of velocity and the direction of acceleration.)

37. [F17-E1] Given three points (x_1,y_1,z_1), (x_2,y_2,z_2), and (x_3,y_3,z_3), the *centroid* of these points is defined to be $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}, \dfrac{z_1+z_2+z_3}{3}\right)$. Find the line which passes through the centroid of the points $(1,4,2)$, $(7,-6,3)$, and $(1,-1,1)$ and is also perpendicular to the plane containing these three points.

38. [F17-E1] The tangent lines to the curve $(3t^3+2t+2, 3-t-3t^2, 4t^3+t^2+3t)$ at times $t=0$ and $t=1$ are intersecting. Find the plane which contains both tangent lines.

39. [F17-E1] The parametric curves $\mathbf{r}(t) = \langle \cos t - \sin t, e^{2t}-1, 2-\ln(2t+1)\rangle$ and $\mathbf{s}(t) = \langle t^3+4t+1, \sin(7t), 2-4\arctan t\rangle$ intersect each other at time $t=0$. Find $\cos\theta$ where θ is the angle between the two tangent lines at time $t=0$.

40. Find the value D so that the plane $2x+y+3z=D$ will *perpendicularly* intersect the curve $\mathbf{r}(t) = \langle 2t^2+8t+1, 8t-7, t^4-8t+1\rangle$.

41. [F18-F] Find the line which is contained in the plane $x+y-3z=7$ and perpendicularly intersects the line $\langle 3+5t, 2-4t, -2+t\rangle$.

42. Find *all* lines which are tangent to the parametric curve $\mathbf{r}(t) = \left\langle t^3-12t+3, t^2+t-4, 2t^2+5t+4\right\rangle$ which *do not* intersect the plane $2x+3y-3z=\pi$.

43. [S19-E1] The curves $\mathbf{r}_1(t) = \langle 5t+2, \ln(t), \sqrt{4-2t-t^2}\rangle$ and $\mathbf{r}_2(s) = \langle 7, \sqrt{2}s^{3/2}-4, e^{s^2-2s}\rangle$ intersect at a point. Find the *plane* that contains the point of intersection and also contains the tangent lines of each curve at the intersection point.

44. [S18-E1] Sphere A has the line segment joining $(0,2,3)$ and $(-4,-2,3)$ as a diameter, and sphere B has the line segment joining $(5,-2,6)$ and $(1,0,4)$ as a diameter. Give the parametric form for the line passing through the *centers* of both spheres.

45. [S18-E1] Given the sphere $x^2+y^2-4y=2-z^2+10z$, and the sphere $x^2+2x+y^2+z^2=0$. Find an equation for the unique line passing through the *centers* of both spheres in parametric form.

46. [S18-E1] Consider the following two intersecting lines:

$$\mathbf{r}(t) = \langle 4,2,3\rangle + t\langle 2,0,3\rangle$$
$$\mathbf{s}(u) = \langle -1,3,-4\rangle + u\langle 1,-1,1\rangle.$$

(a) Find the coordinates of their point of intersection.

(b) Find an equation for the plane containing both lines.

47. [S18-E1] The distance from the point $(4,1,3)$ to a plane is 3. Given that this plane contains the point $(2,-1,2)$, find an equation for the plane. (*Hint: What is the distance between these two points?*)

48. [S18-E1] When a curve intersects a plane the angle of intersection is the angle between the normal vector to the plane and the tangent vector to the curve at the point of intersection.

    Find $\cos\theta$ where θ is the angle of intersection between the curve $\mathbf{r}(t) = \langle \ln t, 2, 3t^2\rangle$ and the plane $3x - y + 2z = 4$ at the point $(0, 2, 3)$.

49. [S18-E1] The curve $\mathbf{r}(t) = \langle t^2 - 2t, t^4 - 4t^2, t^3 - 3t^2 + 2t\rangle$ has intersecting tangent lines at times $t = 0$ and $t = 2$. Find $\cos\theta$ where θ is the angle between these two tangent lines.

50. [F17-E1] Give the equation of the sphere with center $(-1, 2, 3)$ and which is tangent to (touches at one point) the plane $4x - 5y + 2z = 7$.

51. [F17-E1] Give the equation of the sphere with center $(2, -1, 4)$ and which is tangent to (touches at one point) the line $x = 3 + 4t$, $y = 1 + 3t$, $z = 4 + 5t$.

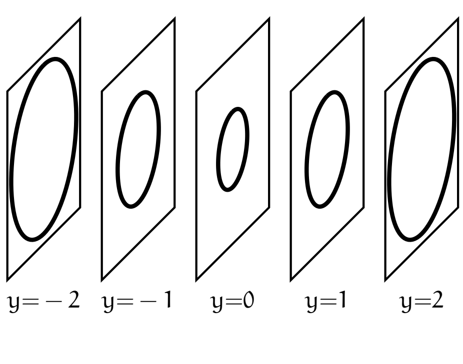
52. Find the plane which contains the points $(2, 4, -1)$ and $(1, 3, -2)$ and *perpendicularly* intersects with the plane $3x + 5y + 4z = 487$.

53. You are flying a small x-wing craft to do a surveillance mission around an asteroid. In order to avoid detection by a nearby imperial cruiser you must always keep the asteroid at the midpoint between your ship and the cruiser. Given that the asteroid is located at $(1, 3, -2)$ and that the path of the cruiser for time $0 \leqslant t \leqslant 1$ is given by $(t^2 + 1, t^3 - t - 1, 4 - e^t)$, determine the path that the x-wing must follow for $0 \leqslant t \leqslant 1$.

54. A particle is traveling along a curve which is parameterized *in cylindrical coordinates* by $r = t^2$, $\theta = t$, $z = 26t - 6t^6$. At time $t = 1$ the particle leaves the curve and moves in a straight line (along its tangent line). Where will the particle collide with the xy-plane if it continues along the tangent line?

55. Shown below are "slices" of a quadric surface taken at a few different values of y. Use this to identify the type of quadric surface and circle your answer from the following list:

    - Ellipsoid
    - Hyperboloid of one sheet
    - Hyperboloid of two sheets
    - Cone
    - Elliptic paraboloid
    - Hyperbolic paraboloid



$y=-2$ $y=-1$ $y=0$ $y=1$ $y=2$

56. Find the point on the line $x = 2 + t$, $y = 3 - 2t$, $z = 3t$ closest to the point $(10, -1, \frac{4}{9})$.

57. Find the centers and radii of *both* spheres which satisfy the following: the sphere is tangent to the planes $2x + y - 2z = 7$ and $8x - 4y + z = -23$, and the center of the sphere lies on the line $x = 2 + t$, $y = 3 + 2t$, $z = -3t$.

58. The curves $\mathbf{r}_1(t) = \langle 5t + 2, \ln(t), \sqrt{4 - 2t - t^2}\rangle$ and $\mathbf{r}_2(s) = \langle 7, \sqrt{2}s^{3/2} - 4, e^{s^2 - 2s}\rangle$ intersect at a point. Find $\cos\theta$ where θ is the *angle* of intersection (the angle between their respective tangent lines).

59. Find the plane that contains the points $(1, 0, 3)$, $(0, 2, 1)$ and *never* intersects with the line $x = 14 - t$, $y = 27 - 2t$, $z = 3t - 137$.

60. [F19-E1] Find the point in the plane $x + 2y - 3z = 1$ which is closest to the point $P = (0, -3, 7)$, *and* find the shortest distance from P to the plane.

61. [F19-E1] Consider the following two lines.

$$\begin{array}{l|l} x = 1 + 2t & x = 5 + 2s \\ y = 2t & y = 1 - s \\ z = -2 + 3t & z = 2 + s \end{array}$$

    (a) Show these two lines intersect by finding the *point* of intersection, and verifying this is a point on *both* lines.

    (b) Find the equation of the plane containing *both* lines.

62. [F19-E1] Consider the surface $\rho \sin^2\phi \cos(2\theta) = \cos\phi$. (Recall that $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.)

    (a) Rewrite this expression in Cartesian coordinates.

    (b) Rewrite this expression in cylindrical coordinates.

    (c) The surface is a quadric surface. Identify the type of quadric surface.

63. You have constructed a track which follows along the parametric curve $\langle t^3, 6t^2 - 18, 9 - 3t\rangle$ for $-5 \leqslant t \leqslant 5$; away from the track there is a cable located on what corresponds to the z-axis. You send a car with a laser pointer attached down the track; as the car moves along the track, the laser pointer sends a beam of light along *the tangent line* (in both directions). Determine the coordinates corresponding to the point(s) on the cable which will be hit by the laser beam as the car goes down the track.

    Hint: points on the z-axis are of the form $(0, 0, z)$.