Week 3 problem bank

1. A particle moving through three dimensional space has $\mathbf{v}(t) = \langle \sec^2 t, 2\sec t \tan t, \tan^2 t\rangle$ as its velocity function. If at $t = 0$ the particle is at $\langle 0, 1, 2\rangle$ find the position of the particle at $t = \frac{\pi}{4}$.
2. Find $\displaystyle\int_0^1 \left(e^t\mathbf{i} + \cos(\pi t)\mathbf{j} - \frac{t}{t^2+1}\mathbf{k}\right) dt$.
3. Find the distance a particle travels along the curve $\left(e^t, e^{-t}, \sqrt{2}\,t\right)$ from $t = 0$ to $t = 1$.
4. Find the distance traveled (i.e., length) along the curve $\left(\frac{2}{5}t^{5/2} - \frac{2}{3}t^{3/2} + 37, \pi^2 - t^2, \frac{1}{2}t^2 + t - 1\right)$ between $t = 0$ and $t = 3$.
5. For $t \geqslant 0$ a particle moves along the curve $\langle 4t^{3/2}, t^2 - 3t, 3\sqrt{3}t\rangle$. Given that starting from time $t = 0$ the particle has traveled a total distance of 40, determine the particle's current location.
6. Find $a_{\mathbf{T}}$ for $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$. Simplify your answer.
7. Find $a_{\mathbf{N}}$ for $\mathbf{r}(t) = \langle e^t + 17, 2e^{-t} - \pi, 37 - 2t\rangle$. Simplify your answer.
8. For $t > 0$, let $\mathbf{r}(t) = \left\langle \ln t, 2t, t^2\right\rangle$. Determine t when acceleration is perpendicular to motion.
9. Find curvature $\kappa(t)$ for $\mathbf{r}(t) = \langle \sin t, \cos t, \frac{1}{2}t^2\rangle$. Simplify your answer.
10. Find the curvature at time $t = 0$ for $\mathbf{r}(t)=$ $\left\langle 71 + \ln(t+1), \tan t + 2t^2 + \sqrt{\pi}, e^t\cos t - \dfrac{e^\pi}{6}\right\rangle$.

Week 3 additional practice

The following problems are drawn from a variety of sources, including previous exams and reviews. These are good practice to prepare for the upcoming exam. The more problems that one is *able* to answer, the better.

1. Find the arc length of the curve $\mathbf{r}(t) = \left\langle e^{2t}, 2te^t - 2e^t + 17, \frac{1}{3}t^3\right\rangle$ for $0 \leqslant t \leqslant 3$.

2. [S17-F] Find the curvature κ for the parameterized curve $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t\rangle$ defined on the interval $0 < t < \pi/2$. Simplify so that no radicals appear in your answer.

3. [F16-F] Consider a particle traveling along the curve

$$\mathbf{r}(t) = t^2\mathbf{i} + \left(t - \frac{t^3}{3}\right)\mathbf{j} + \left(t + \frac{t^3}{3}\right)\mathbf{k}, \qquad t \geqslant 0.$$

(a) Find a unit tangent vector $\mathbf{T}$ in the direction of traveling of $\mathbf{r}(t)$ at $t = 1$.

(b) What is the total distance traveled along the curve by the particle from $t = 1$ to $t = 4$?

4. [F15-F] For the curve $\mathbf{r}(t) = \left(\frac{1}{3}t^3 - 7t\right)\mathbf{i} + (5t^2)\mathbf{j} + \left(\frac{7}{3}t^3 + t\right)\mathbf{k}$, find a_T, i.e., the amount of acceleration pointing in the tangential direction of motion. Simplify your answer to a polynomial function of t.

5. [F15-F] For the curve $\mathbf{r}(t) = \left(\frac{1}{3}t^3 - 7t\right)\mathbf{i} + (5t^2)\mathbf{j} + \left(\frac{7}{3}t^3 + t\right)\mathbf{k}$, find a_n, i.e., the amount of acceleration pointing in the direction which is perpendicular of motion. Simplify your answer to a polynomial function of t.

6. [S15-F] Let $\mathbf{r}(t) = 2t^3\mathbf{i} + 3t^2\mathbf{j} + 3t\mathbf{k}$. Find a_T and a_N in $a = a_T\mathbf{T} + a_N\mathbf{N}$ at $t = 1$. (You don't need to find $\mathbf{T}$ and $\mathbf{N}$.)

7. [F14-F] Consider a particle traveling along the space curve

$$\mathbf{r}(t) = (e^t\cos(t))\mathbf{i} + (e^t\sin(t))\mathbf{j} + 2\mathbf{k}$$

a) Find a unit tangent vector in the direction of travel of $\mathbf{r}(t)$ at $t = \frac{\pi}{2}$.

b) What is the total distance traveled by the particle from $t = 0$ to $t = \frac{\pi}{2}$?

8. [F13-F] Find the arc length of the curve

$$\mathbf{r}(t) = \left\langle e^{2t}, 2te^t - 2e^t + 17, \tfrac{1}{3}t^3\right\rangle,$$

for $0 \leqslant t \leqslant 3$.

9. [F13-F] Find the function for curvature, i.e., $\kappa(t)$, for the curve

$$\mathbf{r}(t) = \left\langle \cos t, \sin t, e^t\right\rangle.$$

Simplify so that the final answer has *no* trigonometric functions.

10. [S16-F] The motion of a particle moving on the xy-plane is described by its position vector $\mathbf{r}(t) = (\cos t + t\sin t, \sin t - t\sin t)$ at time $t \geqslant 0$.

(a) Compute the velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ at time $t \geqslant 0$.

(b) Find the tangential component $a_T(t)$ and the normal component $a_N(t)$ of the acceleration $a(t)$, where $\mathbf{a}(t) = a_T(t)\mathbf{T} + a_N(t)\mathbf{N}$, $\mathbf{T}$ and $\mathbf{N}$ are unit tangent and unit normal vector respectively, at time t.

11. [S12-F] A particle travels along the parametric curve

$$x(t) = t + \ln t, \quad y(t) = t - \ln t, \quad z(t) = 7 - 4\sqrt{t}.$$

Find the distance the particle travels in the time interval $1 \leqslant t \leqslant 2$.

12. Find a_T and a_N for the curve $\langle e^t, 2t, 2e^{-t}\rangle$. Simplify the answer.

13. Find the distance traveled (i.e., length) along the curve $\left(\frac{2}{5}t^{5/2} - \frac{2}{3}t^{3/2} + 37, \pi^2 - t^2, \frac{1}{2}t^2 + t - 1\right)$ between $t = 0$ and $t = 3$. Simplify your answer as much as possible.

14. (a) For $t > 0$, find $\kappa(t)$ for $r(t) = \langle \ln t, 2t, t^2 \rangle$.

    (b) Determine the t which *maximizes* $\kappa(t)$ found in part (a).

    (It suffices to find t, you do not need to prove why it is maximal.)

15. Find the distance traveled along the curve $\left(\frac{1}{5}t^5 - \frac{1}{2}t^4 + 1, \frac{\sqrt{5}}{4}t^4 + 3, \frac{4}{9}t^{9/2} + 7\right)$ between $t = 0$ and $t = 4$. (You do *not* have to simplify after evaluating the integral; but it does simplify.)

16. (a) For $t > 0$, find a_T (i.e., the *amount* of acceleration pointing in the direction of motion) for $r(t) = \langle \ln t, 2t, t^2 \rangle$.

    (b) Use part (a) to determine the t when acceleration is perpendicular to motion.

17. [S19-E1] Find the distance traveled along the curve $\left(2t^2, 7 - \frac{8}{3}t^{3/2}, \frac{2}{5}t^{5/2} - 2t^{3/2} + 1\right)$ between $t = 0$ and $t = 1$.

18. (a) Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ where $\mathbf{r}(t) = \langle e^t + 17, 2e^{-t} + \pi, 31 - 2t \rangle$.

    (b) Determine all times when acceleration will be perpendicular to the direction of motion.

19. You find yourself on an adventure traveling with twelve dwarves to (ahem) borrow Smaug's treasure. As you are crossing the Misty Mountains your company finds itself being chased by goblins on a mountain path. From your vast experience of being chased by Farmer Maggot back in Hobbiton you know that the best chance to evade capture is to hide by getting off the path at a point where the path sharply turns, i.e., at a point with *high curvature* and in particular a point on the road where $\kappa > 10$ would give a point of escape. Glancing at a map of the path (which you fortunately tucked away in your bag) you see that the path on the road can be parameterized by

$$\left\langle 5t^2 - 20t + 37, t^3 - 3t^2 + 41, 3t^2 - 11t + 19 \right\rangle.$$

    [A little known fact is that Hobbiton prides itself on its multivariable calculus educations program.]

    Glóin, one of your dwarf companions, recommends getting off the path corresponding to the point where $t = 2$; Nori, another dwarf, thinks that the road does not bend sharply enough at that point and instead they should only try to outrun the goblins. Which plan of action should you take? Justify your answer.

20. Find the length a particle travels along the curve

$$\mathbf{r}(t) = \left\langle t^2 \sin t, 51 - t^2 \cos t, t^2 + e^{\pi} \right\rangle$$

    for $1 \leqslant t \leqslant \sqrt{8}$.

21. Find the curvature, $\kappa(t)$, for $\mathbf{r}(t) = \langle t + t^2, t - t^2, 2t \rangle$. What is the maximum curvature that this curve attains?

22. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$, for $\mathbf{r}(t) = \langle t + t^2, t - t^2, 2t \rangle$.

23. Find the length a particle travels along the parametric curve $\left(\sin(2t), \frac{1}{2}t^2 - \ln t, \cos(2t)\right)$ for $1 \leqslant t \leqslant e$.

24. [F18-E1] Find the point on the curve:

$$\mathbf{r}(t) = (4 \sin t)\mathbf{i} + (4 \cos t)\mathbf{j} + (3t)\mathbf{k}$$

    that lies a distance of 5π units along the curve from the point $(0, 4, 6\pi)$ in the direction of increasing arc length.

25. [S19-E1] Find the components of acceleration, namely $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$, for the vector valued function $\mathbf{r}(t) = (e^t \cos(t))\mathbf{i} + (e^t \sin(t))\mathbf{j} + t\mathbf{k}$ at time $t = 0$.

26. [F17-E1] A particle's position is given by $\left(\frac{2}{3}\sqrt{3}t^{3/2} + 17, 11 - \frac{2}{5}\sqrt{6}t^{5/2}, \frac{2}{7}t^{7/2} - \frac{2}{3}t^{3/2}\right)$. Find the distance the particle travels between time $t = 0$ and $t = 1$.

27. [F17-E1] A particle's motion is given by $\left(t - \frac{1}{2}\sin(2t) + 11, 2\sin(t) - 2t\cos(t) + 17, \frac{1}{3}t^3 + 2\right)$. Find the distance the particle travels between time $t = 0$ and $t = 3$.
    (Note $1 + \cos(2t) = 2\cos^2 t$ and $1 - \cos(2t) = 2\sin^2 t$.)

28. [F17-E1] Find a_T and a_N (the tangential and normal components of acceleration) for the curve $\mathbf{r}(t) = \langle \frac{2}{3}t^3, t, t^2 \rangle$. Simplify your answer as much as possible.

29. [F17-E1] Find κ (curvature) for the curve $\mathbf{r}(t) = \langle \frac{2}{3}t^3, t, t^2 \rangle$. Simplify your answer as much as possible.

30. Give the equation of the sphere with center $(-1, 2, 3)$ and which is tangent to (touches at one point) the plane $4x - 5y + 2z = 7$.

31. Give the equation of the sphere with center $(2, -1, 4)$ and which is tangent to (touches at one point) the line $x = 3 + 4t$, $y = 1 + 3t$, $z = 4 + 5t$.

32. [S18-E1] A particle's motion is given by $\mathbf{r}(t) = \langle e^t, e^t \cos t, e^t \sin t \rangle$. Find the distance the particle travels between the points $(1, 1, 0)$ and $(e^{\pi/2}, 0, e^{\pi/2})$.

33. [S18-E1] A particle's motion is given by $\mathbf{r}(t) = \langle t - \ln t, 4\sqrt{t}, t + \ln t \rangle$. Find the distance the particle travels in the time interval $1 \leqslant t \leqslant 3$.

34. [S18-E1] Find a_T and a_N (the tangential and normal components of acceleration) for the curve $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} + 2\mathbf{k}$. Simplify your answer as much as possible.

35. [S18-E1] Find the curvature κ for the curve $\mathbf{r}(t) = \langle 3t, 4 \sin t, 4 \cos t \rangle$ at any time t. Simplify your answer as much as possible.

36. [S18-E1] A particle has acceleration $\mathbf{a}(t) = \langle t + 1, e^t, 3t^2 \rangle$. At $t = 1$ the particle has *velocity* $\langle \frac{5}{2}, 0, 1 \rangle$, and at $t = 0$ the particle is at the *position* $\langle -2, 4, 0 \rangle$. Find the *position* of the particle at $t = 2$.

37. [S18-E1] A particle has acceleration $\mathbf{a}(t) = \langle 3e^t, 1, 12t^2 - 4\rangle$. At time $t = 0$ the particle is at $\langle 1, 0, -1\rangle$, and at time $t = 1$ the particle is at $\langle -2, \frac{1}{2}, 0\rangle$. Find the position of the particle at time $t = 2$.

38. (a) Find the cumulative arc length function, $s(t)$, with initial starting point at $t = 0$ for the curve $\mathbf{r}(t) = e^{t^2}\mathbf{i} + \frac{1}{6}t^6\mathbf{j} + (2t^2 - 4)e^{t^2/2}\mathbf{k}$.

    (b) Find $s(3) - s(2)$ and give an explanation of the value.

39. Find the osculating plane to the curve $\mathbf{r}(t)$ at time $t = 0$. (Recall that the osculating plane contains the point on the curve, the direction of velocity, and the direction of acceleration.)

$$\mathbf{r}(t) = \left\langle \tan(t) + 2 + t^3, 1 + t^2, \sin(t) + \cos(t)\right\rangle$$

40. Find κ (curvature) for the curve $\mathbf{r}(t) = \langle \frac{2}{3}t^3, t, t^2\rangle$. Simplify your answer as much as possible.

41. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$, for $\mathbf{r}(t) = \langle t + t^2, t - t^2, 2t\rangle$.

42. Given the velocity of a particle is given by $\langle e^t + 2t, 3t^2 + 1, 2\cos(2t)\rangle$ and at time $t = 0$ the particle is at position $\langle 2, 0, -1\rangle$, then determine the position of the particle at time $t = 2$.

43. Given that the position of a particle is given by $\mathbf{r}(t) = \langle \frac{1}{6}t^6, \frac{2}{5}t^5, \frac{1}{2}t^4\rangle$, find $\mathbf{a_T}$ and $\mathbf{a_n}$ (the tangential and normal components of acceleration).

44. [F19-E1] A particle is traveling through space with acceleration $\mathbf{a}(t) = \langle e^t, -e^{-t}, 4e^{2t}\rangle$ at time t. At time $t = 0$, the particle is at the initial point $(3, 1, 2)$ and has initial velocity $\langle -1, 4, 0\rangle$. Find the position of the particle at time $t = 1$.

45. [F19-E1] Consider the curve $\mathbf{r}(t) = \langle t^2, t - \frac{1}{3}t^3, t + \frac{1}{3}t^3 + 1\rangle$, where $0 \leqslant t < \infty$.

    (a) Find the cumulative arc length function $s(t)$ starting from $t_0 = 0$.

    (b) Find the length of the curve from $(0, 0, 1)$ to $(9, -6, 13)$.

46. [F19-F] Decompose acceleration $\mathbf{a}$ in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ for a particle traveling according to the position function $\mathbf{r}(t) = \langle 2t^2, 2t + \frac{2}{3}t^3, 2t - \frac{2}{3}t^3\rangle$ at $t = 0$ without finding $\mathbf{T}$ and $\mathbf{N}$.