Week 6 problem bank

1. Find all points (x, y) so that $\nabla f(x, y) = \mathbf{0}$ where $f(x, y) = x^2 - 6x + 2y^2 - 10y + 2xy + 42$.
2. On the surface $z = x^3 - 3xy + y^2$ a marble is placed over the point $(1, 2)$. When released the marble initially moves in the direction of steepest decrease. Find a vector $\langle a, b\rangle$ pointing in the direction the marble will initially move relative to the xy-plane.
3. Find the directional derivative for the function $g(x, y, z) = x^2y - 2x^3 + xyz - 6x - 7$ from the point $(-1, 1, 1)$ in the direction of the vector $\langle 6, 3, -6\rangle$.
4. Let $f(x, y, z) = \frac{1}{2}x^2y^3z + \frac{5}{2}z^2$. Find a *unit* vector $\mathbf{u}$ in the direction in which f increases most rapidly at the point $\mathbf{p} = (-2, 1, -1)$, and find the rate of change of f in this direction.
5. Suppose that $f(x, y, z)$ is a differentiable function and at the point $(1, 3, 2)$ that the maximum rate of change of f is 14 and is achieved in the direction starting from the point $(1, 3, 2)$ and going towards the point $(3, -3, 5)$. Find the rate of change of f in the direction starting from the point $(1, 3, 2)$ and going towards the point $(3, 5, 1)$.
6. Find an equation of the tangent plane to the surface $x^2 + xy + y^2 + z^2 = 16$ at the point $(1, 2, 3)$.
7. There are two planes which are *tangent* to the surface $z = x^2 + y^2$ and pass through *both* of the points $(1, 0, 0)$ and $(0, 2, 0)$. One of them is $z = 0$, find the other one. Give your answer in the form $z = Ax + By + C$.
8. An adjustable can machine, has the ability to alter the dimensions of cylindrical cans as they are being produced. The gauges currently read as follows: $h = 5$, $\frac{dV}{dt} = -6\pi$, $\frac{dr}{dt} = -\frac{1}{2}$, and $\frac{dh}{dt} = 1$, where V, h, r are volume, height and radius. The gauge for r is broken. Find the possible value(s) for r.
9. Given the implicit relationship $z + e^z = xy$ find $\dfrac{\partial^2 z}{\partial x \partial y}$ expressed only in terms of z.
10. Suppose that $f(x, y)$ is differentiable and satisfies $f(t^3 - t + 1, 2 - t^2) = t^4 - 4t^3 + 4t + 6$. Find $f_x(1, 1)$ and $f_y(1, 1)$.

The quiz will consist of two randomly chosen problems (up to small variations; so make sure to learn processes).

Week 6 additional practice

The following problems are drawn from a variety of sources, including previous exams and reviews. These are good practice to prepare for the upcoming exam. The more problems that one is *able* to answer, the better.

1. [F17-F] Compute the directional derivative of $g(x, y, z) = 3x^2z - x + \ln(1 + y^2)$ at the point $(1, 1, 1)$ pointing in the direction towards the point $(7, -1, 10)$. (Simplify as much as possible.)

2. Suppose that $f(x, y, z)$ is a differentiable function and at the point $(1, 3, 2)$ that the maximum rate of change of f is 14 and is achieved in the direction starting from the point $(1, 3, 2)$ and going towards the point $(3, -3, 5)$.

   a) Determine $\nabla f(1, 3, 2)$.

   b) Find the rate of change of f in the direction starting from the point $(1, 3, 2)$ and going towards the point $(3, 5, 1)$.

3. [F17-F] Given that the line $\langle 3, 2, 5\rangle + t\langle 3, 2, -1\rangle$ and the line $\langle 3, 2, 5\rangle + t\langle 6, -2, -5\rangle$ are both tangent to the surface $z = f(x, y)$ at the point $(3, 2, 5)$ and that f is differentiable at that point, find $f_x(3, 2)$ and $f_y(3, 2)$. (Hint: how do you write the tangent *plane* in terms of the partials?)

4. [F16-F] Give the parametric equations for the line which is tangent to the curve of intersection of the level surfaces $x^2y + 2y + xz = 1$ and $x^2 + y^2 + z^2 = 6$ at the point $P(-1, 1, 2)$.

5. [S16-F] Consider the function $f(x, y, z) = x^2 + 3xy - z^2 + 2y + z + 4$.

   (a) Find the direction where the function f increases most rapidly at the point $(1, 1, 0)$. Also compute the rate of increase along that direction.

   (b) Find the directional derivative of f at the point $(1, 1, 0)$ in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

6. [F15-F] Find the *unique* point, (x, y, z), on the surface $x^2 - 2xy + 3y^2 - z = 20$ where the tangent plane is parallel to the lines

$$\begin{aligned} x &= t - 2 \\ y &= t + 3 \\ z &= -4t + 5 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} x &= 3t - 4 \\ y &= 2t - 1 \\ z &= 2t + 7 \end{aligned} .$$

   (Hint: what relationship will the gradient vector have with the lines?)

7. [S15-F] Find equations for (a) the tangent plane and (b) normal line to the surface $x^2y + 2xy - z^2 = 5$ at the point $(1, 2, -1)$.

8. [F14-F] Give the parametric equations of the line which is tangent to the curve of intersection of the level surfaces $xyz = 1$ and $x^2 + 2y^2 + 3z^2 = 6$ at the point $P(-1, 1, -1)$.

9. [F13-F] Find the directional derivative of $g(x, y, z) = x^2y - 3xz + 5y + 7z$ at the point $(-1, -1, -1)$ in the direction which goes from $(-1, -1, -1)$ towards the point $(5, 1, 8)$.

10. [F13-F] The surfaces $xz^4 - x^3y^2 + 2yz = 2$ and $3x^2z - yz^3 - x^2y^2 = 1$ both pass through the point $(1, 1, 1)$. Find the *unique* line in *parametric* form which is tangent to both surfaces at the point $(1, 1, 1)$. (Hint: the direction of this line must be perpendicular to the gradients of the surfaces at $(1, 1, 1)$.)

11. [F13-F] Let $z(t) = f\big(x(t), y(t)\big)$ where $f(x, y) = x^2 + y^2$. Determine $z'(3)$ given the following information:

$$\begin{array}{llll} x(1)=3 & x'(1)=2 & y(1)=1 & y'(1)=2 \\ x(2)=5 & x'(2)=6 & y(2)=7 & y'(2)=0 \\ x(3)=1 & x'(3)=3 & y(3)=2 & y'(3)=4 \end{array}$$

12. [F12-F] Find all points (x, y) at which the tangent plane to the surface $z = x^2 - 3xy + \frac{5}{2}y^2 - 2x + 3y$ is parallel to the plane given by the equation $2x - 4y + z = 6$.

13. [S12-F] Let $f(x, y, z) = x^2y^3z + 5z^2$.

(a) Find a *unit* vector $\mathbf{u}$ in the direction in which f increases most rapidly at the point $\mathbf{p} = (-2, 1, -1)$.

(b) What is the rate of change in this direction?

14. [S12-F] The TARDIS (Time and Relative Dimension in Space) machine can be used by the Doctor to travel through space and time. One of the many amazing properties of the TARDIS is its ability to change its interior dimensions in a continuous manner. Currently the TARDIS is changing the dimensions of one of its cylindrical rooms. Recall the volume of a cylinder is $V = \pi r^2 h$.

(a) Given that V, r, and h are functions of t, use the chain rule to find $\frac{dV}{dt}$.

(b) Given that currently

$$h = 4 \text{ ft}, \quad \frac{dr}{dt} = -1 \frac{\text{ft}}{\text{min}},$$
$$\frac{dh}{dt} = 1 \frac{\text{ft}}{\text{min}}, \quad \frac{dV}{dt} = -15\pi \frac{\text{ft}^3}{\text{min}},$$

find the possible value(s) of r.

15. [F18-F] Given that $f(x, y) = g\big(u(x, y), v(x, y)\big)$ and the following information, determine the tangent plane to $f(x, y)$ at the point $(3, 1)$.

(a,b)	$g(a,b)$	$g_u(a,b)$	$g_v(a,b)$	$u(a,b)$	$u_x(a,b)$
$(1,3)$	-2	3	-1	2	5
$(1,4)$	1	-4	7	2	6
$(3,1)$	-5	2	-8	1	7
$(3,4)$	3	1	2	-5	6

(a,b)	$u_y(a,b)$	$v(a,b)$	$v_x(a,b)$	$v_y(a,b)$
$(1,3)$	4	-3	5	6
$(1,4)$	4	5	9	1
$(3,1)$	2	4	6	3
$(3,4)$	2	9	-5	7

16. Given that $f(s, t) = 2e^{s-3t} + s^2 - 4t$ use the linear approximation of $f(s, t)$ at the point $(3, 1)$ to estimate $f(3.1, 1.1)$.

17. Find the rate of change of $f(x, y, z) = x^2z + \ln(xy^2z^3)$ at $(1, 1, 1)$ in the direction $\langle -4, 7, 4\rangle$.

18. You have somehow found yourself in the middle of a math exam and need to estimate $g(0.97, 2.01, 3.02)$ where $g(x, y, z) = ze^{2x-y} + \sin(xz - 3)$ and for some reason do not have a calculator capable of computing this highly precisely. Use linear approximation to give an estimate for the value $g(0.97, 2.01, 3.02)$.

19. Given that $f(x, y)$ is everywhere differentiable and $f(t^2 - 2t, e^t - t) = e^{3t} - \sin(2t) + t^2$, then there is one point (a, b) for which $f_x(a, b)$ can be determined exactly and a different point (c, d) for which $f_y(c, d)$ can be determined exactly. Determine these points, and the corresponding partial derivatives.

20. The point $(0, 1, 2)$ is on the (implicit) surface

$$\sin(2x)e^{(xz-2xy)} + \ln(y)\cos(x^{13}z^7) + (3y)^y \arctan(z-2) = 0.$$

Find the tangent plane to this surface at the point $(0, 1, 2)$. Give your answer in the form $ax + by + cz = d$. (It might be useful to recall $\arctan(0) = 0$ and $\frac{d}{du}\big(\arctan(u)\big) = \frac{1}{1+u^2}$.)

21. Find the directional derivative of the function $g(x, y, z) = x^2y - 2x^3 + xyz - 4x - 7$ at the point $(-1, 1, 1)$ in the direction $\langle 6, 3, -6\rangle$.

22. You are given that the functions $f(x, y)$, $g(x, y)$ and $h(x, y)$ are differentiable everywhere and the following information on tangent planes:

(a,b)	tangent plane to f at (a,b)
$(1,2)$	$4x + 5y - 3z = 7$
$(-5,0)$	$x - 2y + z = 6$
$(3,5)$	$x - y + z = 2$

(a,b)	tangent plane to g at (a,b)
$(1,2)$	$2x - 2y - z = -5$
$(-5,0)$	$3x - 5y + 2z = 8$
$(3,5)$	$4x - 3y - 2z = 11$

(a,b)	tangent plane to h at (a,b)
$(1,2)$	$3x + y - z = 0$
$(-5,0)$	$x + 3y + 7z = 11$
$(3,5)$	$2x + 6y - z = 2$

Find the tangent plane of $k(x, y) = f\big(g(x, y), h(x, y)\big)$ at $(x, y) = (1, 2)$. Give your answer in the form $ax + by + cz = d$. (Hint: you don't need all of the information in the table above.)

23. The set of points (x, y, z) for which $\nabla g(x, y, z)$ is perpendicular to $\nabla g(0, 0, 0)$ for

$$g(x, y, z) = e^{(x-2z)} + \sin(y + 3z) + xy + 4yz + x - 4y + \pi^7$$

is a plane (this is not usually the case, but this function is a Math 265 special). Give the equation for the plane in the form $ax + by + cz = d$. (Remember vectors are perpendicular if and only if their dot products are zero.)

24. Find the tangent plane to the surface $ye^{xz} + yz^2 = 2xy^3 + 2$ at the point $(0, 1, 1)$.

25. Find the rate of change of the function $f(x, y, z) = x^2y + 3xz - 5yz^3 + \pi$ at the point $(1, -1, 1)$ in the direction pointing towards the point $(-3, 6, 5)$.

26. Find the tangent plane to the surface $xy^2 + e^{(x-z)} + \sin(y + 2z) = 5$ at the point $(1, -2, 1)$

27. Consider the implicit function of z given by $x^4 + y^2 + z^6 = 3xz$.

(a) Find expressions for $\frac{\partial z}{\partial z}$ and $\frac{\partial z}{\partial y}$. (These can involve x, y, and z.)

(b) Find the tangent plane to the surface of this function at $(1, 1, 1)$.

28. Find the directional derivative of $g(x, y, z) = x^2 y - 5xz + yz + z^2$ from the point $(1, 2, -1)$ in the direction of the point $(-1, -4, 8)$.

29. Consider the surface given by $x^2 z + 2yz^2 = 3xy^2$.

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(1, 1)$.

(b) Find the tangent plane to the surface at the point $(1, 1, 1)$.

30. Find the directional derivative of $h(x, y, z) = 3e^{x+2y} - 4\sin(z) + 2x - 7y + 8xz + 67$ at the origin in the direction of the point $(-4, 8, -1)$

31. Let $f(x, y) = 5x + 3y - x^2 y + 2xy$. In which direction, given as a unit vector in the inputs, should one initially move from the point $(1, 2)$ to have the largest rate of decrease?

32. Consider the elliptic paraboloid $z = x^2 + y^2$.

(a) Find the tangent planes at the points $(x_0, y_0) = (1, 2)$ and $(x_0, y_0) = (2, -1)$.

(b) Find the unique point (x^*, y^*) so that the tangent plane at (x^*, y^*) intersects both of the tangent planes found in part (a) perpendicularly.

(Hint: two planes intersect perpendicularly whenever the dot products of their normal vectors is 0.)

33. A machine is being set up to make cones to be used for storage. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ where r is a radius and h is a height. The machine ideally makes cones with $r = 3$ and $h = 5$ for a total volume of 15π units, however there tends to be slight imperfections. Given that the volume of a cone must be within π units (i.e., $\Delta V \approx \pm\pi$) and that the height of the cone is set up so that $\Delta h \approx \pm 0.2$ units, then estimate the tolerance that we need to have for the radius (i.e., find Δr).

34. [F18-E2] You are hiking in the "3-D Forest", where your elevation above sea level z, measured in cubits, is given by a differentiable function $f(x, y)$ – where (x, y) represents a sea-level grid position measured in furlongs.

You stop at the point $(3, 2, 20)$ and notice that you are standing in the middle of a small stream. From that point, the stream flows in the direction $\langle -3, 4 \rangle$ where the elevation is dropping at a rate of 15 cubits/furlong.

Since your feet are beginning to get cold, you step out of the stream in the direction $\langle -2, -2 \rangle$. As you take that step, are you climbing or descending? At what rate (include units)?

35. [F18-E2] Suppose that $w = f(x, y, z) = \cos(\pi x) - x^2 y + e^{xz} + yz$. Find an equation of the tangent plane to the $w = 4$ level surface of $f(x, y, z)$ at the point where $x = 0$ and $y = 1$.

36. [S19-E2] Find the directional derivative of $h(x, y) = xy - e^{x-2y}$ at the point $(2, 1)$ in the same direction as the vector $\langle -4, 3 \rangle$.

37. [S19-E2] (a) Find the tangent plane to the surface $x^2 + 3xz + \sin(2x + y) - \ln(xz) = 4$ at $(-1, 2, -1)$.

(b) Estimate the value of x that satisfies $x^2 + 3xz + \sin(2x + y) - \ln(xz) = 4$ when $y = 2.02$ and $z = -0.96$ by using the tangent plane from (a).

38. [S19-E2] Given that $g(x, y) = f\big(f(x, y), f(y, x)\big)$ and the following information about $f(s, t)$, determine the tangent plane to $g(x, y)$ at the point $(2, 1)$.

(s,t)	$(1,2)$	$(1,3)$	$(2,3)$	$(2,1)$	$(3,1)$	$(3,2)$
$f(s,t)$	3	−1	0	1	2	4
$f_s(s,t)$	1	2	3	−1	−2	0
$f_t(s,t)$	−1	−3	5	3	1	3

39. [F18-F] Find the rate of change of $f(x, y, z) = 3x^2 yz^3 - \ln|xyz|$ at $(1, -1, 1)$ in the direction $\langle 4, 5, 3 \rangle$.

40. [S19-F] Find the tangent plane to the surface $yz - x\sin(2 - y) - ze^{(x-1)} = 3$, at the point $(1, 2, 3)$.

41. [F17-E2] Find the value of a so that the tangent plane to the surface $z = x^3 + y^3 + ax^2 + 2y^2 + 2$ at the point $(1, 2)$ goes through the origin (the point $(0, 0, 0)$).

42. [F17-E2] Find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ given that

$$e^{su} + 3u\cos(t) - st = 1.$$

43. [F17-E2] Find $\frac{\partial v}{\partial p}$ and $\frac{\partial v}{\partial q}$ given that

$$qe^{2v} + v^2 p + 2\sec(pq) = 2.$$

44. [F17-E2] Find the tangent plane at the point $(-2, 1, 0)$ to the surface

$$2e^z x^2 + 4xy^3 = y\arctan z.$$

45. [F17-E2] Find the tangent plane at the point $(0, 1, -2)$ to the surface

$$3y^2 \sin x + 4e^{2y+z} = z^2 \cos x.$$

46. Find the tangent plane to $z = f(x, y) = x^3 + 2xy + y^2 + 3\sin(y - 1) + 6e^{x-1}$ at $(x, y) = (1, 1)$.

47. [S18-E2] Let $f(x, y)$ be a differentiable function. Given that at the point $(1, 1)$ the directional derivative in the direction from the point $(1, 1)$ to the point $(4, 5)$ is 8 and that the directional derivative in the direction from the point $(1, 1)$ to the point $(-3, 4)$ is 1, find $f_x(1, 1)$ and $f_y(1, 1)$. Simplify as much as possible.

48. [S18-E2] Let $f(x,y) = xe^{-xy}$. Find all directions $\mathbf{u}$ so that $D_{\mathbf{u}}f(2,0) = 1$.

49. [S18-E2] Find the tangent plane at the point $(1,0,1)$ to the implicitly defined surface:

$$\sin(\pi xz) + yze^{zx^2} = 0$$

50. [S18-E2] The equation $xe^{xz} + z\tan(x+y) = 0$ implicitly defines z as a function of x and y. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

51. [S18-E2] Let $f(x,y)$ be a differentiable function and $g(x,y) = f(x^2y, xy^3)$. Given that at the point $(1,1)$ the tangent plane to $g(x,y)$ is $z = 7x + y - 4$, find the tangent plane at $(1,1)$ to $f(x,y)$. (*Hint:* $g(x,y) = f(u,v)$ *where* $u = x^2y$, $v = xy^3$. *Use the chain rule!*)

52. [S18-E2] Let $f(x,y)$ be a differentiable function and $g(x,y) = f(2x-y, xy^2)$. Given that at the point $(1,1)$ the tangent plane to $g(x,y)$ is $z = 7x - y + 3$, find the tangent plane at $(1,1)$ to $f(x,y)$. (*Hint:* $g(x,y) = f(u,v)$ *where* $u = 2x - y$, $v = xy^2$. *Use the chain rule!*)

53. [S17-F] Find the directional derivative of the function $h(x,y) = e^{-x-y}$ at the point $P_0(\ln 2, \ln 3)$ in the direction of $\mathbf{v} = \langle 1,2 \rangle$. Also, find the directions of maximum increase and maximum decrease at P_0. (Recall that a direction must be a unit vector.)

54. Find a and b so that the planes $z = x + y - 2$ and $z = 4x - 2y - 14$ are both tangent to the surface $z = \frac{1}{a}x^2 + \frac{1}{b}y^2$.

55. Find the tangent plane to the (level) surface $x^2y + 3xe^{2z-4} = 2z\sin(y+3)$ at $(1,-3,2)$.

56. Let $g(x,y,z)$ be a differentiable function. We have the following:

- The rate of change from the point $(-3,4,-2)$ towards the point $(-3,7,-2)$ is 6.
- The rate of change from the point $(-3,4,-2)$ towards the point $(-9,4,6)$ is 6.
- The rate of change from the point $(-3,4,-2)$ towards the point $(9,7,2)$ is 6.

Determine the *maximum* possible rate of change at the point $(-3,4,-2)$.

57. Let $f(x,y) = x^2y + 3y^2 + 4x - 5y$. Determine the rate of change of the function from the point $(5,1)$ in the direction of the point $(2,5)$.

58. Find $\frac{\partial z}{\partial x}$, $\frac{\partial x}{\partial y}$, and $\frac{\partial y}{\partial z}$ for $yz^2 + 3xe^y - 3z\sin(2x) = 6$.

59. Given that $z = -x + 7y + 4$ is tangent to $z = g(3x - 2y, 5x + 3y - 7)$ at $(1,1)$, then find, if possible, the tangent plane to $z = g(x,y)$ at $(1,1)$.

60. Given $f(x,y) = \sin(3x+4y) + xy - x^2 + 6y + 15$, use the linear approximation to $f(x,y)$ at $(-4,3)$ to estimate $f(-3.98, 3.03)$.

61. A plane is tangent to the surface $z = \frac{1}{3}x^2 + \frac{1}{2}y^2$ and perpendicularly intersects with the line $\langle 4+2t, 3-2t, -1-t \rangle$. Determine where the line and plane intersect.

62. Use the linearization of $f(x,y) = xye^{x-2} + 4\sin(y+1) - 3y\ln(x^2-3)$ at $(2,-1)$ to estimate $f(2.1,-1.1)$.

63. [F19-E2] Let $f(x,y) = 2x^2 - 3xy + y^2 + y$, find the directional derivative of f at $(2,-1)$ in the direction of the the vector $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

64. [F19-E2] Find the tangent plane to the surface $z + 1 = xe^{3y}\cos(7z)$ at $(1,0,0)$.

65. [F19-E2] Consider the paraboloid $f(x,y) = x^2 + y^2$.

(a) Sketch the level curve corresponding with $f(x,y) = 2$.

(b) Compute $\nabla f(-1,1)$, and draw this vector in the sketch from (a) where the starting point of the vector is at the point $(-1,1)$.

66. [F19-E2] Use the linearization of the function

$$f(x,y) = \sqrt{2x+3y} - \frac{x}{y}$$

at the point $(3,1)$ to estimate the value of $f(2.97, 1.02)$.

67. [F19-F] Consider the function $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$.

(a) Find the tangent plane to $f(x,y)$ at the point $(3,2,f(3,2))$.

(b) Use the tangent plane from part (a) to approximate the value of the function f at the point $(2.75, 2.2) = (11/4, 11/5)$.