Week 7 problem bank

1. Find the second order Taylor polynomial for $f(x, y) = e^{x+y^2} + x \sin y$ at the point $(0, 0)$.
2. Find the second order Taylor polynomial for $f(x, y) = xy^3$ at the point $(1, 1)$.
3. Find the critical points for
$$g(x, y) = 2x^3 - 2x^2y + 6xy + y^2 - x^2 + 37.$$
4. Find and classify the critical points for
$$f(x, y) = x^3 - 8xy + 2y^2 - 3x + 4y - 23.$$
5. Find and classify all of the critical points for the function $f(x, y) = e^y(y^2 - x^2)$.
6. Let $h(x, y) = x^2 - 6x \cos y$, where x is arbitrary and $-\pi < y < \pi$. Find and classify all critical points.
7. You work for a company which produces Yops and Zans. Currently Yops sell for three dollars each and Zans sell for nine dollars each. The cost to produce y Yops and z Zans is $10 + \frac{1}{2}y^2 + \frac{1}{3}z^3 - yz$ dollars. Determine how many Yops and Zans respectively should be made in order to maximize profit.
8. You run a non-profit organization which produces water filters (w) and x-ray machines (x). Given your resources you can make a combination of w and x that satisfy $3x^2 - 2xw + w^2 = 144$. Given that a water filter helps improve one thousand lives and an x-ray machine helps improve three thousand lives, determine how you should choose w and x to maximize the number of lives that are helped.
9. You have been hired to paint three large non-overlapping circles (red, blue, and green) on the side of a building. The sizes of these circles was left to you, however the total area must be 200π m^2. The contract gives you $3x$ gold bars for painting the red circle of radius x, $4y$ gold bars for the blue circle with radius y, and $5z$ gold bars for the green circle with radius z. What radii should you choose for the various circles to maximize your gold bars?
10. Use the technique of Lagrange multipliers to find the maximum value of $f(x, y) = xy + y$ given that $9x^2 + 10y^4 = 9$.

The quiz will consist of two randomly chosen problems (up to small variations; so make sure to learn processes).

Week 7 additional practice

The following problems are drawn from a variety of sources, including previous exams and reviews. These are good practice to prepare for the upcoming exam. The more problems that one is *able* to answer, the better.

1. Find the maximum and minimum value of $f(x, y) = x^2 + 4y^2$ for closed and bounded region of points satisfying $5x^2 + 12xy + 20y^2 \leqslant 64$.
2. [F17-F] Find the absolute maximum and minimum values of the function $f(x, y) = 2x^2 - 4xy + 4y^2 - 2x$ over the triangular region with vertices at $(0, 0)$, $(2, 2)$ and $(2, -2)$.
3. [S18-F] Find all values for k so that $f(x, y) = x^2 + kxy + y^2$ has a local minimum at (0,0). Give your answer in the form of an interval.
4. Let $f(x, y) = 3x^2 + 4y^3 - 12xy + 2$.

   (a) What are the critical points of $f(x, y)$.

   (b) Classify each of the critical points found in part (a) as local maxima, local minima or saddle points.
5. Consider the function $f(x, y) = 4x^2 + y^4 - 7y^2 - 4xy + 4y - 8x + 10$. Find all critical points of the function f and classify them.
6. [F15-F] Find and classify all three critical points of $h(x, y) = 4x^3 - 6x^2y - 3y^2 + 12y - 7$.
7. [F15-F] Find and classify all three critical points of $h(x, y) = 6xy^2 + 4y^3 - 15x^2 + 36x - 17$.
8. [S15-F] Find all critical points of $f(x, y) = 2x^2 - 2x^2y + y^2$ and classify each of the critical points as local maxima, local minima, or saddle points.
9. [F14-F] Let $f(x, y) = x^2 + 2y^3 + 3y^2 + 2xy$.

   (a) What are the critical points of $f(x, y)$?

   (b) Classify each of the critical points found in part (a) as local maxima, local minima, or saddle points.
10. [F13-F] Find the maximum and minimum value of $f(x, y) = 2x + 3y + 2$ given that $2x^2 + 5xy + 4y^2 = 28$.
11. [F12-F] Find and classify all of the critical points for the function $f(x, y) = x^4 - 4xy - 7x^2 + 4y^2 + 4x - 8y + 20$.
12. You are part of an expedition to Mars. An emergency evacuation caused by a severe storm resulted in you being mistakenly left behind and presumed dead. You realize that it will be years before anyone will be able to come help you, and you know that *the only way you will be able to survive is if you MATH 265 the heck out of your situation*. A first priority for you is to work on growing food. As the team's botanist you have brought some tomato seeds for experimenting and you find some potatoes that had been sent as part of a thanksgiving meal. You have also been able to create a makeshift patch of soil that can be used for growing crops.

    You know that you have extremely limited resources and so you want to make sure to use as little water as possible and still be able to grow fourteen plants (the number you need to sustain you nutritionally). In particular, you note that if you plant p potato and t tomato plants then the amount of water used will be $3t^2 - 3tp + p^2$. **Use the method of Lagrange multipliers** to find the optimum number of potato and tomato plants that should be grown to minimize water usage.

13. Find and classify all of the critical points for $f(x,y) = x^4 - 2x^2y^2 + y^4 + \frac{8}{3}y^3 - 16y^2$.

14. Find and classify all of the critical points for $h(x,y) = 2x^2y - xy^2 - 2x^2 + x + 8$.

15. You are part of an expedition to Mars. An emergency evacuation caused by a severe storm resulted in you being mistakenly left behind and presumed dead. You realize that it will be years before anyone will be able to come help you, and you know that *the only way you will be able to survive is if you MATH 265 the heck out of your situation*. A first priority for you is to work on growing food. As the team's botanist you have brought some tomato seeds for experimenting and you find some potatoes that had been sent as part of a thanksgiving meal. You have also been able to create a makeshift patch of soil that can be used for growing crops.

    You know that there are advantages to crop diversity and so you want to plant the right combination of plants. In particular if there are p potato and t tomato plants then the total yield for the production will be $4tp + p^2$ (note potatoes are essential for your tomato production to help ensure soil nutrition). On the other hand you only have enough "fertilizer" to plant a total of twelve plants. **Use the method of Lagrange multipliers** to find the optimum number of potato and tomato plants that should be grown to maximize yield.

16. Find and classify the critical points of $f(x,y) = x^3 - 3xy^2 + 2y^3 + 6y^2 - 24y + 42$.

17. After crossing the Misty Mountains with your group of dwarf friends you find yourself stopping by the house of Beorn who has offered valuable advice on the best route to take through Mirkwood Forest. In addition to the advice, Beorn has offered some bows for the company to take with them on their journey. It is well known that dwarves are far more efficient at the use of battle axes but bows are much lighter and easier to carry and could effectively be incorporated into their gear. A quick calculation shows that if you plan to carry x axes and y bows that you must have $20x - xy + 12y = 165$. (Actually you could have less, but dwarves refuse to travel with even one bit of unused space; however they have learned from painful experience the problem of carrying excessive weapons and **never travel with more than twenty of any one single kind**.) Beorn has warned of the possibility of large spiders as you cross Mirkwood Forest and so you want to maximize the group's ability to dispose of spiders. In particular since dwarves are more adept at battle axes, then with x axes and y bows the group can take on as many as $3x + y$ spiders.

    Being the junior companion you have been tasked with determining the loads to maximize the group's abilities to fight spiders. How much of each weapon should the group go with?

    (Hint: $119 = 7\cdot 17$ and $175 = 5\cdot 35$)

18. The tangent plane gives an approximation of the function nearby, the second order Taylor polynomial gives an even better approximation! Find the second order Taylor polynomial of $f(x,y) = 3xe^y + 2ye^x - 4xy$ around the point $(0,0)$ and use this to approximate $f(0.1, 0.2)$.

19. Find and classify the critical points of $f(x,y) = x^2y + x^2 - 6xy + y^2 - 6x + 7y + 47$.

20. Write the second order Taylor polynomial for the function $f(x,y) = 2e^{x+y} - \sin(xy)$ around the point $(0,0)$.

21. Find and classify the critical points for $g(x,y) = xy^2 + 2xy + \frac{1}{2}x^2 + 31$.

22. You have made it into the championship episode of *The Ultimate Extreme Happy Fun Roller Coaster Happy Fun Hour*. You are now in the third round, the "blackjack round" where you have to be the fastest in time to gain exactly 21 points where points are earned by riding on two different roller coasters, the *"X"treme tumbler* and the *"Y" won't it stop*. Because the roller coasters take different lengths of time to ride the points are strangely distributed. Namely, if you ride the "X" x times and the "Y" y times then your total score will be $2xy - 3x$. As you watch the contestants going before you, you can't help but notice they are dazed as they switch the rides slowing them down. In particular you observe the total amount of time the contestants ride the two coasters satisfy $2x + 3y + xy$.

    Using the method of Lagrange multipliers, determine the optimal number of times to ride the two roller coasters to gain 21 points as fast as possible.

23. Find and classify the critical points for the function $g(x,y) = x^2y + xy^2 - 3xy$.

24. You have made it into the championship episode of *The Ultimate Extreme Happy Fun Roller Coaster Happy Fun Hour*. You are now in the first round where you have a total of thirty minutes to gain points where points are earned by riding on two different roller coasters, the *"X"treme tumbler* and the *"Y" won't it stop*. Because the roller coasters take different lengths of time to ride the points are strangely distributed. Namely, if you ride the "X" x times and the "Y" y times then your total score will be $3xy - 5y$. As you watch the contestants going before you, you can't help but notice they are dazed as they switch the rides slowing them down. In particular you observe the total amount of time the contestants ride the two coasters satisfy $4x + 3y + xy$.

    Using the method of Lagrange multipliers, determine the optimal number of times to ride the two roller coasters to gain the maximum number of points.

25. [F18-E2] Let $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

    (a) Find all the critical points of $f(x,y)$.

    (b) Classify each of the critical points found in part (a) as local maxima, local minima, or saddle points.

26. [F18-E2] Kelly is in the business of producing different types of widgets and gadgets. For every g gadgets and w widgets produced, Kelly's profit is given by $P(g, w) = 30g^{1/3}w^{2/3}$. Each gadget costs \$2 to produce and each widget costs \$1 to produce. If Kelly only has \$300 to spend on production, how many gadgets and widgets should be produced to maximize profit?

27. [S19-E2] Let $f(x, y) = \dfrac{1}{x} + xy + \dfrac{1}{y}$ where $x \neq 0$ and $y \neq 0$.

    (a) Find the critical point(s) of $f(x, y)$.

    (b) Classify each critical point as a local maximum, local minimum, or a saddle point.

28. [S19-E2] Find the absolute maximum and minimum of $f(x, y) = 2x^2 + xy - 5x + \frac{1}{2}y^2 - 2y + 4$ on or inside the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.

29. [F18-F] Determine the absolute minimum and maximum values of the function $f(x, y) = 2x^2 - 2xy + y^2 - y + 3$ on the closed triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$.

30. [S19-F] (a) Find the second order Taylor polynomial for $f(x, y) = e^{5x} \ln(1 + y)$ at $(0, 0)$.

    (b) Suppose that $g(x, y)$ is a function having second order Taylor polynomial $P(x, y) = 5 - x + 3y + 2x^2 - 4xy + y^2$ at $(1, -1)$. Find the value of $g_x(1, -1)$.

31. [S19-F] *Use the method of Lagrange multipliers* to find the maximum value of $f(x, y) = x^{2/3}y^{1/3}$ subject to the constraint $x + 4y = 96$.

32. [F17-E2] Find and classify the critical points of $f(x, y) = \frac{2}{3}x^3 - x^2 + x^2y - 2xy - y^2 - 9y$.

33. [F17-E2] Find and classify the critical points of $f(x, y) = 2x^2y - 2xy + xy^2 - y^2 + \frac{4}{3}x^3 - \frac{1}{2}x^2 - 6x$.

34. [F17-E2] Use the method of **Lagrange multipliers** to find the maximum of $2x - y$ given that $3x^2 - 4xy + 2y^2 = 6$. Also give the point (x, y) where the maximum is achieved. (The curve $3x^2 - 4xy + 2y^2 = 6$ is a closed and bounded set so the maximum must be achieved.)

35. [F17-E2] Use the method of **Lagrange multipliers** to find the minimum of $4x^2 - 3xy + 2y^2$ given that $2x + 5y = 12$. Also give the point (x, y) where the minimum is achieved. (You do not have to prove that it is a minimum.)

36. Find the second order Taylor polynomial of $f(x, y) = e^{3x} \cos(2y)$ at the point $(0, 0)$.

37. Use the method of *Lagrange multipliers* to find the maximum and minimum possible values, and the corresponding locations where they are achieved for the function $f(x, y) = 2x + 3y$ given that $x^2 + xy + y^2 = 84$.

38. Let $g(x, y) = 7 + x + 3y^2 + 4xy + 6x^3 + xy^2 - 5x^2y$ be the *third* order Taylor polynomial to $f(x, y)$ *centered at* $(2, 1)$. Use this to determine all the second order derivatives for $f(x, y)$ at $(2, 1)$.

39. [S18-E2] Find and classify all critical points of $f(x, y) = x^3 + 2xy + y^3$.

40. [S18-E2] Find and classify all critical points of $f(x, y) = \frac{2}{3}x^3 + 2x^2 - x^2y - 4yx + \frac{1}{3}y^3 - 5y + 7$.

41. [S18-E2] Find the maximum and minimum values of $f(x, y) = xy$ given that $16x^4 - 32xy + y^4 = 256$.

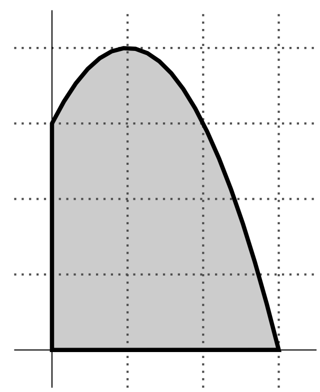
42. [S18-E2] Find the maximum and minimum values of $f(x, y) = xy$ given that $x^4 + 4xy + y^4 = 6$.

43. [F16-F] Let $f(x, y) = 3x^2 + 4y^3 - 12xy + 2$. Find and classify the critical points of $f(x, y)$.

44. Find the second order Taylor polynomial for $h(x, y) = \ln(xy) + 2x^2y + e^{2x-3y+1}$ at $(1, 1)$.

45. Find and classify all critical points of $g(x, y) = 4x^2y - 2xy^2 + 7x^2 + 8x$.

46. Let R be region bounded by the lines $y = 0$, $x = 0$, and $y = -x^2 + 2x + 3$ with corners at $(0, 0)$, $(0, 3)$, and $(3, 0)$ as shown below. Find the minimum and maximum values of $f(x, y) = x^2 - 2x + y^2 - 4y + 3$ on R.



47. Find the maximum and minimum values of $f(x, y) = xy$ given that $x^2 - 4xy + 9y^2 = 30$.

48. Use the method of *Lagrange multipliers* to find potential locations of local maximums and minimums of the function $f(x, y) = 2\sqrt{2}x + y$ given that $x^4 + x^2y^2 = 48$.

49. [F19-E2] Find and classify the critical points of $h(x, y) = \frac{1}{3}x^3 - x^2y + 18y^2 - 72y - 24$. (Hint: when determining the sign of an expression, pull out common factors to simplify.)

50. [F19-F] Use the method of **Lagrange multipliers** to find the maximum and minimum values of $f(x, y) = y - x$ given that $x^2 + 2xy + 3y^2 = 3$.

51. [F19-E2] Find the absolute (global) maximum and minimum values of

$$f(x, y) = y^2 - x^2 - y + x$$

in the closed triangular region bounded by the lines $x = 0$, $y = 0$ and $y = 1 - x$.