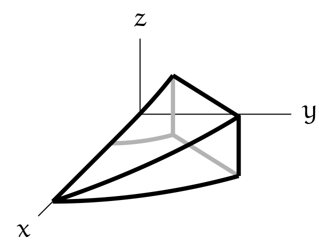
Week 10 problem bank

1. Convert the integral $\int_0^\pi \int_0^1 \int_0^{\sqrt{3}r} r^2 \sin\theta \, dz \, dr \, d\theta$ from cylindrical coordinates to Cartesian coordinates with order $dz\,dy\,dx$.
2. Convert the integral $\int_0^\pi \int_0^1 \int_0^{\sqrt{3}r} r^2 \sin\theta \, dz \, dr \, d\theta$ from cylindrical coordinates to spherical coordinates.
3. Set up, but do not evaluate, an integral in spherical coordinates to calculate the volume of the region inside the surfaces $x^2+y^2+z^2=9$, below the surface $z=\sqrt{x^2+y^2}$ and above the plane $z=0$.
4. Evaluate the integral $\int_0^1 \int_{1-y}^{4-y} y e^{\sqrt{x+y}} \, dx \, dy$ by making the change of variables $u=\sqrt{x+y}$ and $v=y$. (Note $\int ue^u \, du = (u-1)e^u + C$.)
5. Compute the integral $\int_0^1 \int_{-y^3}^{1-y^3} (x+y^3)e^{y(x+y^3)} \, dx \, dy$ by carrying out the following change of variables $u=y$ and $v=x+y^3$.
6. Find $\int_0^2 \int_{x-x^2}^{2-x^2} 6x\cos\left((x^2+y)^3\right) dy \, dx$ by making the substitutions $u=x^2+y$ and $v=x$.
7. Given $\mathbf{F}=\langle y\sin x, x^2+2z, xy+e^z\rangle$, find div $\mathbf{F}$.
8. Given $\mathbf{F}=\langle y\sin x, x^2+2z, xy+e^z\rangle$, find curl $\mathbf{F}$.
9. Evaluate
$$\int_0^1 \int_y^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 2yz \, dz \, dx \, dy$$
by converting to spherical or cylindrical coordinates.
10. Evaluate
$$\int_0^1 \int_x^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z^2 \, dz \, dy \, dx$$
by converting to spherical coordinates.

Week 10 additional practice

The following problems are drawn from a variety of sources, including previous exams and reviews. These are good practice to prepare for the upcoming exam. The more problems that one is *able* to answer, the better.

1. [S18-F] Evaluate $\int_1^2 \int_x^{2x} 2e^{x^2} \, dy \, dx$ **by making the change of variables** $u=x$ and $v=\frac{y}{x}$.
2. [F17-F] Let $\mathcal{D}$ be the solid with $0 \leqslant y \leqslant x$, $1 \leqslant x^2+y^2 \leqslant 9$, and $0 \leqslant z \leqslant \arctan(\frac{y}{x})$ (see picture; note $\arctan(\frac{y}{x})$ has a simple expression in cylindrical coordinates). Given $\mathcal{D}$ has density function $\delta(x,y,z)=3z$, find the total mass of $\mathcal{D}$.

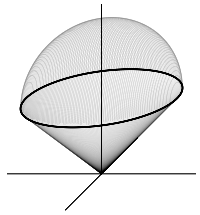


3. Find the integral of the function $f(x,y,z)=(x^2+y^2+z^2)^{-3/2}$ over the region between the spheres of radius 2 and 4, centered at the origin.
4. [F17-F] Compute the integral
$$\int_1^2 \int_{x^2}^{8-x^2} 8xe^{((y+x^2)^2-4(y+x^2))} \, dy \, dx$$
by first carrying out the following change of variables: $u=y+x^2$ and $v=y-x^2$.
(Hint: the region in the uv-plane will be a triangle.)
5. [F16-F] Change the integral $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} (x^2+y^2) \, dz \, dy \, dx$ into the following coordinates. (Do not evaluate the integral.)
(a) Cylindrical coordinates.
(b) Spherical coordinates.
6. [S16-F] Let $\mathbf{D}$ be the hemispherical solid region described by $x^2+y^2+z^2 \leqslant 1$, $z \geqslant 0$. If the mass density at the point (x,y,z) is given by $\delta(x,y,z)=4\sqrt{x^2+y^2+z^2}$, compute $\bar{z}$, the z coordinate of the center of mass. (*Hint: use spherical coordinates.*)
7. [S15-F] Find the center of mass of the solid region D lying above the cone $z=\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2=4$, with density function $\delta(x,y,z)=\sqrt{x^2+y^2+z^2}$.
8. [F14-F] Find the center of mass of the solid between the sphere $\rho=\cos\phi$ and the upper hemisphere of $\rho=2$ if the density function is $\delta(x,y,z)=z$.
9. [F13-F] Compute the integral
$$\int_0^1 \int_{-y^3}^{1-y^3} (x+y^3)e^{y(x+y^3)} \, dx \, dy$$
by carrying out the following change of variables: $u=y$ and $v=x+y^3$.
10. [F13-F] Let $\mathcal{S}$ be the solid region satisfying
$$z \geqslant 0, \quad 1 \leqslant x^2+y^2 \leqslant 4, \quad \text{and} \quad z \leqslant \sqrt{x^2+y^2-1}$$
(i.e., the area inside a cylinder of radius 2, above the xy-plane and "outside" the hyperboloid of one sheet $x^2+y^2-z^2=1$). Given that $\delta(x,y,z)=2z$, find mass of the solid.

11. [F12-F] Set up an iterated integral using spherical coordinates to compute the mass of the solid inside the sphere $x^2+y^2+z^2=4z$ and below the surface $z=\sqrt{x^2+y^2}$ with mass density $\delta(x,y,z)=(x^2+y^2)$. **No need to evaluate this integral.**

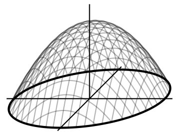
12. [S12-F] Let T be the solid which consists of all the points satisfying $0 \leqslant x^2+y^2 \leqslant 4$ and $0 \leqslant z \leqslant 3+\sqrt{4-x^2-y^2}$. (This solid is a cylinder of radius 2 and height 3 topped off with a hemisphere.) Given that for this solid $\delta(x,y,z)=1$ find the center of mass. (You can use symmetry and volumes of solids to simplify the calculations.)

13. Set up (but do not evaluate) an integral in spherical coordinates to find the mass of the object which lies above the cone $z=\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+(z-1)^2=1$ (see picture below) given the density function $\delta(x,y,z)=\sqrt{x^2+y^2}$.



14. Find $\displaystyle\int_0^1\int_{y^3}^{2-y^3} \frac{y^2}{1+(x+y^3)^2}\,dx\,dy$ by making the substitution $u=x+y^3$ and $v=x-y^3$.

15. Let S be the solid given by $0 \leqslant z \leqslant 1-x^2-y^2$ (see picture below). Determine the value of $\alpha>0$ so that if the density is $\delta(x,y,z)=z^\alpha$, then S has mass $\frac{1}{30}\pi$.



16. Find the mass and centroid of the solid consisting of points $\sqrt{x^2+y^2} \leqslant z \leqslant \sqrt{1-x^2-y^2}$ with density function $\delta(x,y,z)=z$.

17. Rewrite (but do NOT evaluate) the following integral

$$\int_0^1\int_{x^2+x}^{x^2+1} \frac{6x}{1+(y-x^2)^3}\,dy\,dx$$

by using the change of variables $u=y-x^2$ and $v=x$.

18. Rewrite (but do NOT evaluate) the following integral

$$\int_0^1\int_{y-y^2}^{y} \frac{1}{(1+x-y+y^2)^2}\,dx\,dy$$

by using the change of variables $u=y$ and $v=x-y+y^2$.

19. Find the mass and the centroid of the solid consisting of points $\sqrt{x^2+y^2} \leqslant z \leqslant 1$ with density function $\delta(x,y,z)=1-\sqrt{x^2+y^2}$.

20. Find the center of mass for the solid bounded above by the surface $z=4$, outside the cylinder $x^2+y^2=1$, and bounded below by $z=x^2+y^2$ which has density $\delta(x,y,z)=2$.

21. Use the substitution $u=x^2+y$ and $v=x^2-y$ to rewrite the following sum of integrals as a *single* iterated integral in terms of u and v. Do not attempt to evaluate the resulting integral.

$$\int_0^1\int_{-x^2}^{x^2} \frac{4xe^{x^2-y}}{1+(x^2+y)^2}\,dy\,dx + \int_1^{\sqrt{2}}\int_{x^2-2}^{2-x^2} \frac{4xe^{x^2-y}}{1+(x^2+y)^2}\,dy\,dx$$

22. Rewrite the following integral in terms of cylindrical coordinates and then evaluate:

$$\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{\sqrt{4-y^2-z}} \frac{1}{4-x^2-y^2}\,dx\,dz\,dy.$$

23. Use the substitutions $u=x$ and $v=y+x^3$ to rewrite the following integral in terms of u and v and use this to calculate the integral

$$\int_0^1\int_{-x^3}^{x-x^3} 20(y+x^3)^3 e^{x^5}\,dy\,dx.$$

24. Consider the solid region which consists of points inside a sphere of radius 1 centered at the origin and satisfying $z \geqslant 0$. Given the density $\delta(x,y,z)=z^2$ for this solid, determine the center of mass of this solid. (Use symmetry where possible; it might be useful to recall that in cylindrical coordinates that $z=z$ and in spherical coordinates $z=\rho\cos\phi$.)

25. Suppose that the density of a solid consisting of the sphere of radius 1 centered at the origin and satisfying $z \geqslant 0$ has density $\delta(x,y,z)=z^\alpha$ where $\alpha>-1$. When α is close to -1 the density is heavily amassed at the bottom and $\bar{z}\approx 0$ and as $\alpha\to\infty$ the density at the bottom gets small and the density accumulates near the top so $\bar{z}\approx 1$. Determine the *unique* $\alpha>-1$ so that $\bar{z}=\frac{1}{2}$.

26. Rewrite (but do NOT evaluate) the following integral

$$\int_0^{\sqrt{1/2}}\int_{y^2}^{1-y^2} \frac{8xy+8y^3}{1+(x-y^2)^3}\,dx\,dy$$

by using the change of variables $u=x-y^2$ and $v=x+y^2$.

27. Willy Wonka is introducing a new candy called "The Black Hole". This is a dark chocolate confection which is in the shape of a sphere with a diameter of 2 centimeters where the center is hollow, i.e., The Black Hole is a sphere of chocolate with a smaller sphere "punched out" in the center. In keeping with his over the top tradition Wonka has perfected a method to get the density of chocolate to be $1/\rho$ grams per cubic centimeter where ρ is the distance to the center (yes, his machine can achieve infinite density at a point, a true black hole!). Wonka has decided that The Black Hole should have a weight of 4 grams. Find the thickness of the shell of The Black Hole that Wonka will need in order to achieve this goal. (In other words find the difference in the radii of the outer sphere and the inner sphere.)

28. Rewrite (but do NOT evaluate)
$$\int_1^4 \int_{\sqrt{y-1}}^{\sqrt{y}} x e^{y-x^2}\,dx\,dy$$
using the change of variables $u = y$ and $v = y - x^2$.

29. Willy Wonka has decided to introduce the MiFLLe Lozenge (MiFLLe for "Melts Fast and Last Long"). The shape will be a cylinder with a diameter of 2 centimeters and a height of $\frac{1}{2}$ centimeter. Wonka has also retuned his machine so that the densest part is through the axis of rotation of the lozenge and decreases as we move out from the center, in particular the density is $2 - \beta r$ grams per cubic centimeter where r is the distance to the center (i.e., the axis of rotation) and β can be set by a dial on the machine. If Wonka wants to have the lozenge have exactly two grams of mass, then what should be the setting for β?

30. Rewrite (but do NOT evaluate)
$$\int_1^4 \int_{e^x}^{e^x+1} \frac{\sin(y - e^x)}{x^2}\,dy\,dx$$
by using the change of variables $u = y - e^x$ and $v = x$.

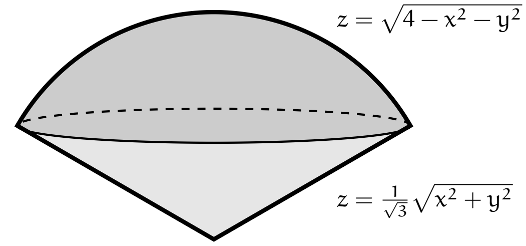
31. [F18-E3] (a) Calculate the divergence of the following vector field
$$\begin{aligned} F = {} & \big(x\cos(e^{y^2}) + 3\big)\mathbf{i} \\ & + \big(x\sqrt{z^2+1} + \arctan(xz+1)\sin(y)\big)\mathbf{j} \\ & + \big(e^{2y}\sin(3x) + z^3\big)\mathbf{k}. \end{aligned}$$
(b) Calculate the curl of the following vector field
$$F = (\sin(yz) - x)\mathbf{i} + (\cos(xz) - y)\mathbf{j} + (e^{xy} - z)\mathbf{k}$$

32. [F18-E3] Convert but **DO NOT EVALUATE** the following integral
$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \int_1^{\sqrt{4-x^2-y^2}} (x^2+y^2)\,dz\,dx\,dy$$
(a) into cylindrical coordinates;

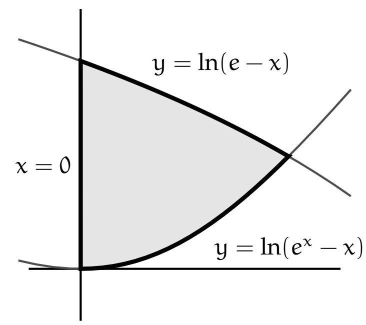
(b) into spherical coordinates.

33. [F18-E3] Evaluate $\displaystyle\int_1^2 \int_0^{2-y} 2y\cos\big((x+y)^3 - 3(x+y)\big)\,dx\,dy$ **by first making the change of variables** $u = x + y$ and $v = y^2$ (for reference the region of the integral in the xy-plane is shown below).

34. [S19-E3] Find the mass of the solid with $\frac{1}{\sqrt{3}}\sqrt{x^2+y^2} \leqslant z \leqslant \sqrt{4-x^2-y^2}$ and $\delta(x,y,z) = 5z\sqrt{x^2+y^2}$. (Hint: look at the shapes involved in this problem.)



35. [S19-F] Find $\displaystyle\int_0^1 \int_{\ln(e^x - x)}^{\ln(e-x)} e^y \sin\big((x+e^y)\big(\ln(x+e^y) - 1\big)\big)\,dy\,dx$ *by first carrying out the change of variables* with $u = x$ and $v = x + e^y$. (For reference, the region is sketched below.)



36. [S17-F] Find the integral of the function $f(x,y,z) = (x^2+y^2+z^2)^{3/2}$ over the region between the spheres of radius 2 and 4, centered at the origin.

37. Find the center of mass of the upside down cone $0 \leqslant z \leqslant 1 - \sqrt{x^2+y^2}$ with $\delta(x,y,z) = (x^2+y^2)^2$.

38. Find the mass of the three dimensional solid where $z \geqslant 0$, $x^2+y^2 \geqslant 1$, $x^2+y^2+z^2 \leqslant 2$, and with density $\delta(x,y,z) = \dfrac{2}{x^2+y^2+z^2}$.

39. Find $\displaystyle\int_{-1}^0 \int_0^{\sqrt{y+1}} \frac{6}{1+(x^2-y)^3}\,dx\,dy$ by carrying out the change of variables $v = x^2$ and $u = x^2 - y$.

40. Find both divergence *and* curl for the following functions.

(a) $\mathbf{F} = \cos(y)e^x\mathbf{i} + x^2z^3\mathbf{j} + (y + \tan(z))\mathbf{k}$

(b) $\mathbf{G} = \big\langle \ln(x^2+y^2+1), \arctan(y) + z^3, z\sin(xy)\big\rangle$

41. Find the curl and divergence for the vector field $\mathbf{H} = (x^3 + ze^y)\mathbf{i} + (\ln(y^2+1) - 2xz^2)\mathbf{j} + (5xyz)\mathbf{k}$.

42. Convert the following integral in cylindrical coordinates to an integral in spherical coordinates. Do NOT evaluate.

$$\int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\sqrt{2-r^2}}^{r} z^2 \, dz\, dr\, d\theta$$

43. A solid is described by the pair of equations $1 \leqslant x^2+y^2 \leqslant 4$ and $0 \leqslant z \leqslant \sqrt{x^2+y^2}$. Given that the the density of the solid satisfies $\delta(x,y,z) = 3+z$, determine its total mass.

44. [F19-E3] Consider the vector field $\mathbf{F}(x,y,z) = \langle xyz, y+z^2, x-y^3z\rangle$.

    (a) Compute $\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$.

    (b) Compute $\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$.

45. [F19-E3] Let R be the triangle with vertices $(0,0)$, $(2,1)$ and $(3,0)$. Using the change of variables $u = 2y-x$, $v = x+y$, rewrite the integral $\iint_R (x+y)\sin(2y-x)\, dA$ as an integral in terms of u and v, including bounds. **Do not evaluate the integral.**

46. [F19-E3] Use cylindrical coordinates to compute the moment of inertia I_z about the z-axis of the solid cone D of radius 1 and height 1 given by the equation

$$\sqrt{x^2+y^2} \leqslant z \leqslant 1$$

    and with constant density $\delta(x,y,z) = 2$.

47. [F19-F] A solid is formed by the set of points with $z \geqslant 0$ and $x^2+y^2+z^2 \leqslant R^2$ (the upper half of a solid sphere of radius R). Given the density of the solid is $\delta(x,y,z) = 3z^3$ and the mass of the solid is 16π, determine R.