

Chapter 7

Probability

7.1 Experiments, Sample Spaces, and Events

- A **(random) experiment** is an activity with observable results.
- The **sample space** S of an experiment is the set of all outcomes.
- Each outcome of an experiment is called a **sample point**.
- Any subset of the sample space is called an **event**. In other words, an event is a collection of some sample points.

Ex. Toss a coin and observe whether it falls heads or tails. The sample space $S = \{H, T\}$. H and T are sample points. All possible events are

$$\emptyset, \quad \{H\}, \quad \{T\}, \quad \{H, T\} = S.$$

Let E and F be two events. There are 3 basic operations of events (Fig 1, p.355, Venn diagram).

1. The **union** of E and F is the event $E \cup F$.
2. The **intersection** of E and F is the event $E \cap F$.
3. The **complement** of E is the event $E^c = S - E$.

We also use $E - F := E \cap F^c$.

Two events E and F are **mutually exclusive** if $E \cap F = \emptyset$ (Venn diagram).

Ex. Let E , F and G be three events. Use \cup , \cap , and c to describe the following events:

- The event that both E and F occur.

- The event that E but not F occurs.
- The event that E occurs but neither of the events F or G occurs.

Ex. (HW 2, p.359) Let $S = \{a, b, c, d, e, f\}$ be a sample space. Let $E = \{a, b\}$, $F = \{a, d, f\}$, and $G = \{b, c, e\}$ be events. Find the events $F \cup G$ and $F \cap G$.

Ex. Select a letter randomly from the words “BUSINESS CALCULUS”.

1. What is the sample space S of this experiment?
2. Describe the event “the letter selected is a vowel”.

HW. P7.1: SC 1, 2, Ex 1, 3, 5, 15, 17, 19, 33, 35

7.2 Definition of Probability

Repeat an experiment independently for N times. Suppose that an event E occurs M times in N trials. Then M/N is the **relative frequency** of the event E . When $N \rightarrow \infty$, M/N may approach a limit $P(E)$, called the **empirical probability** of E .

Ex. (HW 12, p.367) Survey 500 adults living with children for how many days a week they cook at home. The results are:

Number of days	0	1	2	3	4	5	6	7
Respondents	25	30	45	75	55	100	85	85

Determine the empirical probability.

Now consider an experiment with only finitely many outcomes, say the sample space $S = \{s_1, \dots, s_n\}$. Each $\{s_i\}$ is called a **simple event**. These simple events are mutually exclusive. The table of **probability distribution** gives the probability $P(s_i)$ for each simple event $\{s_i\}$ (See Table 3, p.363.) The **probability function** P satisfies the following properties:

1. $0 \leq P(s_i) \leq 1$, $i = 1, 2, \dots, n$.
2. $P(s_1) + P(s_2) + \dots + P(s_n) = 1$.
3. For any event $E = \{s_{i_1}, s_{i_2}, \dots, s_{i_k}\}$, $P(E) = P(s_{i_1}) + P(s_{i_2}) + \dots + P(s_{i_k})$.

Ex. Flip a fair coin twice and count the number of heads appearing. Give the probability distribution.

Def. A finite **uniform sample space** is a sample space $S = \{s_1, s_2, \dots, s_n\}$ of an experiment in which the outcomes are equally likely:

$$P(s_1) = P(s_2) = \dots = P(s_n) = \frac{1}{n}.$$

Ex. Find the following probability distributions for a standard deck of 52 cards:

1. (HW2, p.367) Record the suit (S,H,C,D) of a randomly selected card.
2. Record the rank (Ace, 2, 3, ..., King) of a randomly selected card.

Ex. (Applied Ex 4, Table 5, p.366, Testing New Product)

HW. P7.2: SC 1, 2, Ex 5, 7, 9

7.3 Rules of Probability

Let S be a sample space of an experiment. The probability function P assigns to each event E a number $P(E)$ in $[0, 1]$.

Thm 7.1 (Properties of Probability Function). *Let E and F be events of the sample space S .*

1. $0 \leq P(E) \leq 1$.
2. $P(E^c) = 1 - P(E)$.
3. $P(S) = 1, P(\emptyset) = 0$.
4. $P(E \cup F) + P(E \cap F) = P(E) + P(F)$ (Venn diagram).
5. In particular, if E and F are mutually exclusive ($E \cap F = \emptyset$), then (Venn diagram)

$$P(E \cup F) = P(E) + P(F).$$

6. If E_1, \dots, E_n are mutually exclusive events, then (Venn diagram)

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = P(E_1) + P(E_2) + \dots + P(E_n).$$

Ex. (Applied Ex 1, p.372, SAT Verbal Scores)

Ex. (HW 8, p.377) Select a card from a 52-card deck. What is the probability of the event “a diamond or a king is drawn”?

Ex. (HW 26, p.377) $P(E) = .6, P(F) = .4,$ and $P(E \cap F) = .2$. Compute

$$a.P(E \cup F) \quad b.P(E) \quad c.P(F^c) \quad d.P(E^c \cap F)$$

Ex. (HW 47, p.380, optional) The probability that Bill can solve a problem is p_1 and the probability that Mike can solve it is p_2 . Show that the probability that Bill and Mike working independently can solve the problem is $p_1 + p_2 - p_1p_2$.

HW. P7.3: SC 1, 2 Ex 5, 7, 33, 37

7.4 Use of Counting Techniques in Probability

Thm 7.2 (Uniform Sample Space). *Let E be a uniform sample space and E any event. Then*

$$P(E) = \frac{\# \text{ of outcomes in } E}{\# \text{ of outcomes in } S} = \frac{|E|}{|S|}.$$

Thm 7.3. *If an event E can be obtained by k steps, in which step 1 has n_1 choices, step 2 has n_2 choices, ..., step k has n_k choices, then the number of sample points in E is*

$$|E| = n_1 n_2 \cdots n_k.$$

Ex. (Ex 1, p.381) *An unbiased coin is tossed six times. What is the probability that the coin will land heads*

a. *Exactly three times?* b. *At most three times?* c. *On the first and the last toss?*

Ex. (HW 12, p.386) *Four balls are selected at random without replacement from an urn containing three white balls and five blue balls. Find the probability of the event that two or three of the balls are white.*

Ex. (HW19, Quality Control, p.386) *Two light bulbs are selected at random from a lot of 24, of which 4 are defective. What is the probability that*

1. *Both of the light bulbs are defective?*
2. *At least 1 of the light bulbs is defective?*

Ex. (HW36, p.387) *If a 5-card poker hand is dealt from a well-shuffled deck of 52 cards, what is the probability of being dealt a straight (but not a straight flush)?*

HW. P7.4: SC 1, 2, EX 3, 5, 9, 13

7.5 Conditional Probability and Independent Events

I. Conditional Probability

Let A and B be events of a sample space S . If event A already happens, the probability that event B happens may be different.

Ex. (*Ex1, p.388*) Two cards are drawn without replacement from a 52-card deck.

1. What is the probability that the first card drawn is an ace?
2. What is the probability that the second card drawn is an ace given that the first card drawn was not an ace?
3. What is the probability that the second card drawn is an ace given that the first card drawn was an ace?

The **conditional probability of B given A** is denoted by $P(B|A)$. See (Fig 14, p.389) for a Venn diagram explanation of the conditional probability.

Thm 7.4. *If A and B are events in an experiment and $P(A) \neq 0$, then the conditional probability of B given A is*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Ex. (*HW2, p.398*) Let A and B be two events in a sample space S such that $P(A) = .4$, $P(B) = .6$, and $P(A \cap B) = .3$. Find a. $P(A|B)$ b. $P(B|A)$.

Thm 7.5 (Product Rule). $P(A \cap B) = P(A) \cdot P(B|A)$.

Ex. (*Ex 5, p.392*) Two cards are drawn without replacement from a 52-card deck. What is the probability that the first card drawn is an ace and the second card drawn is a face card?

II. Tree Diagrams

A **finite stochastic process** is an experiment consisting of a finite number of stages in which the outcomes and associated probabilities of each stage depend on those of the preceding stages. **Tree diagrams** are useful in analyzing such processes.

Ex. (*Applied Ex 6, Quality Control, p.394*)

Ex. (*Applied Ex 7, Quality Control, p.394*)

III. Independent Events

Def. Two events A and B are **independent events** if the happening of one event does not affect the probability of the other event, that is,

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B).$$

Since $P(A|B) = \frac{P(A \cap B)}{P(B)}$. We have

Thm 7.6 (Test for Independence of Two Events). Two events A and B are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

If A and B are independent, then each pair of A and B^c , A^c and B , A^c and B^c , are independent resp.

Ex. (HW10, p.398) If A and B are independent events, $P(A) = .35$ and $P(B) = .45$, find
a. $P(A \cap B)$ b. $P(A \cup B)$.

Ex. (HW16, p.399) A pair of fair dice is rolled.

$E =$ "the number falling uppermost on the first die is 4",

$F =$ "the sum of the numbers falling uppermost is 6". Compute

a. $P(F)$ b. $P(E \cap F)$ c. $P(F|E)$ d. $P(E)$ e. Are E and F independent events?

Thm 7.7. If events E_1, E_2, \dots, E_n are independent, then

$$P(E_1 \cap E_2 \cap \dots \cap E_n) = P(E_1) \cdot P(E_2) \cdot \dots \cdot P(E_n).$$

Ex. If events A, B, C are independent and $P(A) = .2$, $P(B) = .4$, and $P(C) = .5$. Compute:
a. $P(A \cap B \cap C)$ b. $P(A \cap B^c \cap C^c)$.

HW. P7.5: SC 1, 2 EX 1, 3, 5, 7, 11, 23, 37, 41

7.6 Bayes' Theorem

Def. • **A Priori probability:** *The probability that an event will occur.*

- **A Posteriori probability:** *The probability after the outcomes of the experiment have been observed.*

Ex. Three machines A, B, and C, produce similar engine components. Machine A produces 45%, machine B produces 30%, and machine C produces 25%. The defective rate is 6% for machine A, 4% for machine B, and 3% for machine C, respectively. One component is selected at random and is found to be defective. What is the probability that the component selected was produced by machine A?

(See Venn diagram in Fig 21, p.402, and tree diagram in Fig 22, p.403)

Thm 7.8 (Bayes' Theorem). Let A_1, A_2, \dots, A_n be a partition of a sample space S . Let E be an event of S . Then the a posteriori probability

$$P(A_i|E) = \frac{P(A_i) \cdot P(E|A_i)}{P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) + \dots + P(A_n) \cdot P(E|A_n)}, \quad 1 \leq i \leq n.$$

Ex. (HWs 4, p.406) Use the Venn diagram to draw a tree diagram. Find $P(A)$, $P(B)$, $P(C)$, and $P(D)$.

Ex. (HW 8, p.406) Use the tree diagram to find

- a. $P(A) \cdot P(D|A)$ b. $P(B) \cdot P(D|B)$ c. $P(A|D)$

Ex. (Applied Ex 2, p.404, Income distributions) A study in a large metropolitan area about the annual incomes of married couples shows that:

Annual Family Incomes, \$	Married Couples, %	Income Group with Both Spouses Working, %
150K and over	4	65
100K-150K	10	73
75K-10K	21	68
50K-75K	24	63
30K-50K	30	43
Under 30K	11	28

1. What is the probability that a couple selected at random has two incomes?
2. If a randomly chosen couple has two incomes, what is the probability that the annual income of this couple is %150K or more?
3. If a randomly chosen couple has two incomes, what is the probability that the annual income of this couple is greater than 49,999?

HW. 7.6 SC 1, 2, EX 3, 5, 7, 25