Linear Differential Equations Chapter One

Name: Answer

Totally 30 minutes. Please write in details for partial credits.

2. (10 points) The motion of a set of particles moving along the x-axis is governed by the differential equation $\dfrac{dx}{dt} = t^3 - x^3$, where $x(t)$ denotes the position at time t of the particle.

   (a) If a particle is located at $x = 1$ when $t = 2$, what is its velocity at this time?

   $$\text{velocity} = \frac{dx}{dt} = t^3 - x^3 = 2^3 - 1^3 = 7$$

(b) Show that the acceleration of a particle is given by $\frac{d^2x}{dt^2} = 3t^2 - 3t^3x^2 + 3x^5$.

$$\frac{dx}{dt} = t^3 - x^3 \Rightarrow \frac{d^2x}{dt^2} = \frac{d}{dt}(t^3 - x^3) = 3t^2 - 3x^2\frac{dx}{dt}$$
$$= 3t^2 - 3x^2(t^3 - x^3)$$
$$= 3t^2 - 3t^3x^2 + 3x^5$$

(c) If a particle is located at $x = 2$ when $t = 2.5$, can it reach the location $x = 1$ at any later time? [Hint: $t^3 - x^3 = (t - x)(t^2 + xt + x^2)$.]

If the particle can reach $x=1$ at a later time $(t>2.5)$, then its velocity $\frac{dx}{dt} < 0$ since $x = 2 > 1$ at $t = 2.5$.

However, $\frac{dx}{dt} = t^3 - x^3 = (t-x)(t^2+xt+x^2) > 0$ when $t > 2.5$ and $x = 1$.

This is a contradiction. So the particle can not reach $x=1$ at a later time.

3. (10 points) Use Euler's method with step size $h = 0.2$ to approximate the solutions to the initial value problem

$$y' = \frac{1}{x}(y^2 + y), \qquad y(1) = 1$$

at the points $x = 1.2,\ 1.4,\ 1.6$, and 1.8.

$x_0 = 1 \quad y_0 = 1 \quad x_{n+1} = x_n + h, \quad y_{n+1} = y_n + h \cdot f(x_n, y_n)$

Here $f(x,y) = \frac{1}{x}(y^2 + y)$

$x_1 = 1 + 0.2 = 1.2, \quad y_1 = 1 + 0.2 \cdot \frac{1}{1}(1^2 + 1) = 1.4$

$x_2 = 1.2 + 0.2 = 1.4, \quad y_2 = 1.4 + 0.2 \cdot \frac{1}{1.2}(1.4^2 + 1.4) \approx 1.96$

$x_3 = 1.4 + 0.2 = 1.6, \quad y_3 = 1.96 + 0.2 \cdot \frac{1}{1.4}(1.96^2 + 1.96) \approx 2.79$

$x_4 = 1.6 + 0.2 = 1.8, \quad y_4 = 2.79 + 0.2 \cdot \frac{1}{1.6}(2.79^2 + 2.79) \approx 4.11$

Linear Differential Equations Chapter Two

Name: Answer

Time: 50 minutes. Please write in details for partial credits.

1. (10 points) Classify the following equations as *separable*, *linear*, *exact*, or *none of these.*

   (a) $\frac{ds}{dt} = t\ln(s^{2t}) + 8t^2$

   separable

   (b) $s^2 + \frac{ds}{dt} = \frac{s+1}{st}$

   none of these

   (c) $3t = e^t \frac{dy}{dt} + y\ln t$

   linear

   (d) $3r = \frac{dr}{d\theta} - \theta^3$

   linear

   (e) $(ye^{xy} + 2x)dx + (xe^{xy} - 2y)dy = 0$

   exact

2. (10 points) Solve the initial value problem: $\dfrac{dy}{dx} = (1+y^2)\tan x, \qquad y(0)=\sqrt{3}$

$$\frac{dy}{1+y^2} = \tan x \, dx \Rightarrow \int \frac{dy}{1+y^2} = \int \tan x \, dx$$

$$\Rightarrow \tan^{-1} y = -\ln(\cos x) + C$$

By $y(0)=\sqrt{3}$ we get

$$\tan^{-1}\sqrt{3} = -\ln(\cos 0) + C = C$$

$$\Rightarrow C = \frac{\pi}{3}$$

$$\Rightarrow \boxed{y = \tan\left[-\ln(\cos x) + \frac{\pi}{3}\right]}$$

3. (10 points) Obtain the general solution to the equation $\dfrac{dy}{dx} - y - e^{3x} = 0$

$$\frac{dy}{dx} - y = e^{3x} \Rightarrow \text{integrating factor } \mu(x) = e^{\int -1 dx} = e^{-x}$$

$$\Rightarrow \frac{d}{dx}[e^{-x}y] = e^{2x}$$

$$\Rightarrow e^{-x}y = \frac{1}{2}e^{2x} + C$$

$$\Rightarrow \boxed{y = \frac{1}{2}e^{3x} + ce^{x}}$$

6. (10 points) Consider the equation $(y^2 + 2xy)dx - x^2\,dy = 0$.
   (a) Show that this equation is not exact.
   (b) Show that multiplying both sides of the equation by y^{-2} yields a new equation that is exact.
   (c) Use the solution of the resulting exact equation to solve the original equation.
   (d) Were any solutions lost in the process?

(a) $\frac{\partial}{\partial y}(y^2+2xy) = 2y+2x \qquad \frac{\partial}{\partial x}(-x^2) = -2x$ so the equation is $\boxed{\text{not exact}}$

(b) Multiply both sides by y^{-2} $\Rightarrow$ $(1+\frac{2x}{y})dx - \frac{x^2}{y^2}dy = 0$

$\Rightarrow$ Let $M = 1+\frac{2x}{y}$. $N = -\frac{x^2}{y^2}$. then

$\frac{\partial M}{\partial y} = -\frac{2x}{y^2}$, $\frac{\partial N}{\partial x} = -\frac{2x}{y^2}$ so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

and the equation is exact.

(c) $\frac{\partial F}{\partial x} = M \Rightarrow F = \int M\,dx + g(y) = \int (1+\frac{2x}{y})dx + g(y) = x + \frac{x^2}{y} + g(y)$.

$\frac{\partial F}{\partial y} = N \Rightarrow \frac{\partial}{\partial y}(x+\frac{x^2}{y}+g(y)) = -\frac{x^2}{y^2}$

$\Rightarrow -\frac{x^2}{y^2} + g'(y) = -\frac{x^2}{y^2}$

$\Rightarrow g(y) = 0$

So the solution to the diff eq is $\boxed{F(x,y) = x + \frac{x^2}{y} = C}$

(d) Since we multiply both side of the diff eq by y^{-2}. we lost the solution $\boxed{y=0}$ in the process