Chapter 1

Introduction

1.1 Background

Briefly go through:
(p4) dependent variables, independent variables, coefficients, ordinary differential equations, partial differential equations, order of a differential equations. (p5) linear / nonlinear differential equations.

1.2 Solutions and Initial Value Problems

An *nth*-order ordinary equation (with x independent, y dependent) is of the form:

$$F(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}) = 0 \tag{1.2.1}$$

In many cases, we can isolate the highest order term and get

$$\frac{d^n y}{dx^n} = f(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1} y}{dx^{n-1}}) \tag{1.2.2}$$

Def 1.1. A function $\phi(x)$ is called an **explicit solution** to the equation (1.2.1) on an interval I, if $y = \phi(x)$ satisfies the equation for all x in I.

Ex. (p7) ex 2.

Def 1.2. A relation $G(x, y) = 0$ is said to be an **implicit solution** to equation (1.2.1) on the interval I if it defines one or more explicit solutions on I.

Ex. (p8) ex 4, (p9) ex 5.

Def 1.3 (Initial Value Problem). See p10.

Ex. (p11) ex 7.

The simplest initial value problem is

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0.$$

The following important theorem guarantees a unique solution of this problem on a neighborhood of the initial point.

Theorem 1.4 (Existence and Uniqueness of Solution, See Fig 1.5). *Given the initial value problem*

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0,$$

assume that f *and* $\frac{\partial f}{\partial y}$ *are continuous functions in a rectangle*

$$R = \{(x,y) : a < x < b,\ c < y < d\}$$

that contains (x_0, y_0). *Then the initial value problem has a unique solution* $\phi(x)$ *in some interval* $x_0 - \delta < x < x_0 + \delta$, *where* δ *is a positive number.*

Ex. (p13) ex 8, ex 9.

Homework

1.2 (p14-16): 1, 11, 13, 17, 21, 23, 25, 29

1.3 Direction Fields

(p16-21, with figures) For the first order differential equation $\frac{dy}{dx} = f(x,y)$, it is often difficult to find a formula of the implicit or explicit solution. Direction field provides a useful way to visualize the solutions of this problem. It is also useful to display the existence and the uniqueness of the corresponding initial value problems.

Def 1.5. A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve there is called a **direction field** for the differential equation.

Ex. Fig 1.6 (p17) Shows the direction field of $\frac{dy}{dx} = x^2 - y$ and how to get solution curves from the direction field.

Ex. Fig 1.7 (a) is the direction field for $\frac{dy}{dx} = -2y$. You can verify that the solutions are $y = Ce^{-2x}$. Fig 1.8 (a) shows how to solve the corresponding initial value problem.

Ex. Fig 1.7 (b) is the direction field for $\frac{dy}{dx} = -y/x$. You can verify that the solutions are $y = \frac{C}{x}$. Fig 1.8 (b) shows how to solve the corresponding initial value problem.

We can study the properties of a differential equation from a direction field.

Ex. (ex 1, p19) The *logistic equation* for the population p (in thousands) at time t of a certain species is given by $\dfrac{dp}{dt} = p(2-p)$ for some $p > 0$. From the direction field (Fig 1.10, p20), answer the following:

1. If the initial population is 3000 [that is, $p(0) = 3$], what can you say about the limiting population $\lim_{t\to+\infty} p(t)$?

2. Can a population of 1000 ever decline to 500?

3. Can a population of 1000 ever increase to 3000?

- ⋆ Method of Isoclines: Computer softwares (like MAPLE) are often used to sketch the direction fields. Sometimes the **method of isoclines** are helpful in hand sketching the direction fields.

Def 1.6. An **isocline** for the differential equation $y' = f(x, y)$ is a set of points in the xy-plane where all the solutions have the same slope dy/dx; thus, it is a level curve for the function $f(x, y)$.

Ex. (Fig 1.11, p21) Draw direction field of $y' = x + y$ from isoclines.

Homework

1.3 (p22-24): 1, 2, 5, 6, 18

1.4 The Approximation Method of Euler

Euler's method (or the tangent method) is a procedure for constructing approximate solutions to an initial value problems

$$y' = f(x, y), \qquad y(x_0) = y_0$$

The procedure is like sketching a solution curve from direction field by a sequence of small line segments. **Euler's Method** are done by connecting the following points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots$

$$\begin{aligned} x_{n+1} &= x_n + h, \\ y_{n+1} &= y_n + hf(x_n, y_n), \qquad n = 0, 1, 2, \cdots \end{aligned}$$

Here h is the **step length**.

Ex. See ex 1 (p25) and ex 2 (p26).

Euler's method is the local linear approximation of the function. Higher degree approximations, such as **Taylor polynomial** of degree n or **Taylor series**, can do a better work.

Homework

1.4 (p28-29): 4, 5, 10, 11