Chapter 2

First Order Differential Equations

2.1

2.2 Separable Equations

A first order differential equation $\frac{dy}{dx} = f(x,y)$ is called **separable** if

$$f(x,y) = g(x)p(y).$$

That is, $f(x,y)$ can be expressed as a function $g(x)$ that depends only on x times a function $p(y)$ that depends only on y. The differential equaiton can be solved as follow:

$$\frac{dy}{dx} = g(x)p(y) \quad \Longleftrightarrow \quad \frac{1}{p(y)}dy = g(x)dx \quad \Longleftrightarrow \quad \int \frac{1}{p(y)}dy = \int g(x)dx$$

Suppose $\int \frac{1}{p(y)}dy = H(y) + C$ and $\int g(x)dx = G(x) + C$. Then we get the implicit solution

$$H(y) = G(x) + C.$$

⋆ In doing the above operations, we should be cautious about the possible loss of solutions. For example, what happen if $p(y) = 0$?

Ex. (ex 1, p41) Solve the nonlinear equation $\frac{dy}{dx} = \frac{x-5}{y^2}$.

Ex. (ex 2, p43) Solve the initial value problem

$$\frac{dy}{dx} = \frac{y-1}{x+3}, \qquad y(-1) = 0.$$

Ex. (ex 3, p44) Solve the nonlinear equation $\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos y + e^y}$.

Ex. (HW 29, p47) Uniqueness Questions.

Ex. (HW 37, p48) Compound Interest.

Homework

2.2 (p46-48): 1-6, 7, 8, 17, 18, 31, 32

2.3 Linear Equations

In this section, we study the **linear first-order equation** of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

It is first-order, and the coefficients are functions of x. Quotient both sides by $a_1(x)$ (assuming that $a_1(x) \neq 0$). The equation is changed to the standard form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{2.3.1}$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$.

If $P(x) = 0$ or $Q(x) = 0$, the equation (2.3.1) is separable and is easy to solve. For general case, we multiple both sides of (2.3.1) by certain function $\mu(x)$ (to be determined):

$$\mu\frac{dy}{dx} + \boxed{\mu P y} = \mu Q. \tag{2.3.2}$$

The left-hand side of (2.3.2) is similar to the product rule for

$$\frac{d}{dx}[\mu y] = \mu\frac{dy}{dx} + \boxed{\frac{d\mu}{dx}y}$$

Let us choose μ so that $\mu P = \dfrac{d\mu}{dx}$, that is, $\dfrac{1}{\mu}\,d\mu = P\,dx$. Solve the equation. We may choose

$$\mu(x) = e^{\int P(x)\,dx}. \tag{2.3.3}$$

Then (2.3.2) becomes $\dfrac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$. The solution for (2.3.1) is

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + C\right] \tag{2.3.4}$$

Statement 2.1 (Method of Solving Linear Equations).

1. *Write the equation in the standard form*

$$\frac{dy}{dx} + P(x)y = Q(x).$$

2. *Calculate the integrating factor* $\mu(x)$ *by the formula*

$$\mu(x) = \exp\left[\int P(x)\, dx\right].$$

3. *Multiply the equation in standard form by* $\mu(x)$ *and recalling that the left-hand side is just* $\frac{d}{dx}[\mu(x)y]$*, obtain*

$$\begin{aligned} \mu(x)\frac{dy}{dx} + P(x)\mu(x)y &= \mu(x)Q(x) \\ \iff \quad \frac{d}{dx}[\mu(x)y] &= \mu(x)Q(x). \end{aligned}$$

4. *Integrate the last equation and solve for* y*:*

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\, dx + C\right]$$

Ex. (ex1, p51; fig2.5, p52) Solve $\dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{2y}{x^2} = x\cos x, \quad x > 0.$

Ex. (hw22, p55) $(\sin x)\dfrac{dy}{dx} + y\cos x = x\sin x, \;\; y\left(\dfrac{\pi}{2}\right) = 2.$

Ex. (hw16, p44) $(x^2+1)\dfrac{dy}{dx} = x^2 + 2x - 1 - 4xy.$

Ex. (ex2, p52) Radioactive decay of an isotope.

Theorem 2.1 (Existence and Uniqueness of Solution). *If* $P(x)$ *and* $Q(x)$ *are continuous on an open neighborhood* (a,b) *of* x_0*, then for any* y_0*, there exists a unique solution* $y(x)$ *on* (a,b) *to the initial value problem*

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0.$$

The solution is given by (2.3.4) *for a suitable* C*.*

Ex. (ex3, p53, skip if no time) For the initial value problem

$$y' + y = \sqrt{1 + (\cos x)^2}, \;\; y(1) = 4,$$

find the value of $y(2)$.

Homework

2.3 (p54-58): 1-6, 7, 8, 17, 18, 23, 35, 36

2.4 Exact Equations

Suppose that a plane curve is defined by an implicit equation: $F(x,y) = C$, where C is a constant. Taking total differential along the curve gives:

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0.$$

So

$$\frac{dy}{dx} = f(x,y) := -\frac{\partial F/\partial x}{\partial F/\partial y}.$$

Conversely, if a differential equation $\frac{dy}{dx} = f(x,y)$ can be expressed as

$$M(x,y)dx + N(x,y)dy = 0$$

where $M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$ for certain function $F(x,y)$, then

$$M(x,y)dx + N(x,y)dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dF(x,y) = 0$$

and the solution of $\frac{dy}{dx} = f(x,y)$ is given implicitly by $F(x,y) = C$ for an arbitrary constant C.

Ex. (ex1, p59) $\frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y}$.

Def. The differential form $M(x,y)dx + N(x,y)dy$ is said to be **exact** in a rectangle R if there is a function $F(x,y)$ such that

$$\frac{dF}{dx}(x,y) = M(x,y) \quad \text{and} \quad \frac{dF}{dy}(x,y) = N(x,y)$$

for all (x,y) in R. In such case, the equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called an **exact equation**.

If $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$ on the rectangle R, then clearly

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\frac{\partial F}{\partial y} = \frac{\partial N}{\partial x}.$$

The reverse is also TRUE!

Theorem 2.2. *Suppose that both $M(x,y)$ and $N(x,y)$ have continuous first order partial derivatives in a rectangle R, then $M(x,y)dx + N(x,y)dy = 0$ is an exact equation in R if and only if the compatibility condition*

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

holds for all (x,y) in R.

The proof is similar to that of Green's theorem. A constructive proof is given in (p62-63).

⋆ **Method for Solving Exact Equations**

1. If $Mdx + Ndy = 0$ is exact, then $\partial F/\partial x = M$. Integrate the last equation w.r.t. x to get
$$F(x,y) = \int M(x,y)dx + g(y) \tag{2.4.1}$$
2. Take the partial derivative w.r.t. y of both sides of equation (2.4.1) and substitute N for $\partial F/\partial y$. We can solve for $g'(y)$.
3. Integrate $g'(y)$ to get $g(y)$ up to a constant. Substituting $g(y)$ into (2.4.1) gives $F(x,y)$.
4. The solution to $Mdx + Ndy = 0$ is given implicitly by
$$F(x,y) = C.$$

Ex. (ex2, p63) $(2xy - \sec^2 x)dx + (x^2 + 2y)dy = 0.$

Alternatively, we may integrate $N = \partial F/\partial y$ w.r.t. y first, and then solve the equation.
Ex. (ex3, p64) $(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0.$

Ex. (ex4, p65) Show that $(x + 3x^3 \sin y)dx + (x^4 \cos y)dy = 0$ is not exact but that multiplying this equation by x^{-1} yields an exact equation. Solve the equation.

Ex. (HW27, p66)

Homework

Section 2.4 (p65-67): 1-8, 10, 11, 13, 21, 22, 29, 30