Chapter 3

Mathematical Models and Numerical Methods Involving First-Order Equations

3.1 Mathematical Modelling

The process of mimicking reality by using the language of mathematics is known as **mathematical modelling**. There are three basic steps:

1. **Formulate the Problem**.
   Requires an understanding of the problem area as well as the mathematics.

2. **Develop the Model**. Two things to be done:

   (a) Decide which variables are important and which are not. Then classify the important variables as independent variables or dependent variables.

   (b) Determine the relationships that exists among the relevant variables.

3. **Test the Model**.

3.2 Compartmental Analysis

Many complicated processes can be broken down into distinct stages and the entire system modelled by describing the interactions between the various stages. Such system are called **compartmental** and are graphically depicted by **block diagrams**.

Ex. (One-compartment system)

$$\text{Input rate} \longrightarrow \boxed{x(t)} \longrightarrow \text{Output rate}$$

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}$$

3.2.1 Mixing Problems

Ex. (ex1, p96)

Ex. (ex2, p98) [Be aware of the overflow situation].

3.2.2 Population Models

1. **Malthusian model**

   Let $p(t)$ denote the population of a species in a compartment. It is natural to assume that the growth rate $\dfrac{dp}{dt}$ of the population is proportional to the population $p(t)$. This gives the mathematical model (where k = birth rate – death rate):

   $$\frac{dp}{dt} = kp, \qquad p(0) = p_0. \tag{3.2.1}$$

   It is separable and the solution is $p(t) = p_0 e^{kt}$.

   Ex. (ex3, p100) In 1790 the population of the United States was 3.93 million, and in 1890 it was 62.98 million. Using the Malthusian model, estimate the U.S. population as a function of time.

2. **Logistic model**

The death rate of a population may involve a competition within the population, such as malnutrition, inadequate medical supplies, communicable diseases, violent crimes, etc. Thus we get the logistic model

$$\frac{dp}{dt} = k_1 p - k_3 \frac{p(p-1)}{2}$$

or

$$\frac{dp}{dt} = -Ap(p - p_1), \qquad p(0) = p_0. \tag{3.2.2}$$

The equation (3.2.2) has two equilibrium solutions: $p(t) = p_1$ and $p(t) = 0$. The non-equilibrium solutions can be found by separating variables. (Fig 3.4, p102)

$$p(t) = \frac{p_0 p_1}{p_0 + (p_1 - p_0)e^{-Ap_1 t}}.$$

Ex. (ex4, p102) The population of the United States was 3.93 million in 1790, 17.07 million in 1840, and 62.98 million in 1890. Using the Logistic model to estimate the population at time t.

In the above example, we only use the data in 1790, 1840, and 1890, to estimate the parameters p_0, A, and p_1. To make a more robust model, we may use the least square linear fit using all of the data. (See (20) and Fig 3.5 on p103)

Homework

3.2 (p104-107): 1, 3, 9, 10, 13, 14, 15, 18, 19

3.3 Heating and Cooling of Buildings

We will formulate a mathematical model to describe the temperature inside a building. The building may be viewed as a compartment. Its inside temperature $T(t)$ may be affected by several important factors:

1. The heat produced by people, lights, and machines inside the building. They cause a rate of increase/decrease in temperature that represented by $H(t)$.

2. The heating/cooling supplied by the furnace/air conditional. This rate of increase/or decrease in temperature will be represented by $U(t)$.

3. The effect of the outside temperature $M(t)$. By **Newton's law of cooling**, the rate of change in $T(t)$ by outside temperature is proportional to the difference between the outside temperature $M(t)$ and the inside temperature $T(t)$. So the rate of change of $T(t)$ due to $M(t)$ is $K[M(t) - T(t)]$ for some constant K.

Altogether, we set up the following model:

$$\frac{dT}{dt} = K[M(t) - T(t)] + H(t) + U(t)$$

It is a linear system... We solve that

$$T(t) = e^{-Kt}\left\{\int e^{Kt}\left[KM(t) + H(t) + U(t)\right]\, dt + C\right\}. \tag{3.3.1}$$

Ex. (ex1, p108)

In the above example, $1/K$ is called the **time constant for the building**.

Ex. (ex2, p109)

Ex. (ex3, p111)

Homework

3.3 (p113-114): 1, 2, 5, 6, 7, 12

3.4 Newtonian Mechanics

Newton's Law of Motion:

1. When a body is subject to no resultant external force, it moves with a constant velocity.
2. When a body is subject to external force(s), the time rate of changes of the body's momentum is equal to the vector sum of the external forces acting on it.
3. When a body interact with a second body, the force of the first body acting on the second is equal in magnitude, but opposite in direction, to the force of the second body on the first.

Thus we get $m\frac{dv}{dt} = F(t, x, v)$. In this section, we focus on situations where the force F does not depend on x. So we get a first-order equation

$$m\frac{dv}{dt} = F(t, v)$$

Ex. (ex1, p116)

Ex. (ex2, p118)

Ex. (ex3, p119)

Ex. (ex4, p121)

Homework

3.4 (p121-124): 1, 2, 6, 11, 12, 14, 20