Chapter 4

Linear Second-Order Equations

4.1 Introduction

4.2 Homogeneous Linear Equations

The general form of a second-order constant coefficient differential equation:

$$ay'' + by' + cy = f(t), \qquad a \neq 0.$$

The special case $f(t) = 0$ is called a homogeneous linear differential equation:

$$ay'' + by' + cy = 0 \tag{4.2.1}$$

If both $y_1 = f(x)$ and $y_2 = g(x)$ are solutions of the homogeneous differential equation (4.2.1), then every linear combination $y_3 = C_1 f(x) + C_2 g(x)$ is also a solution of (4.2.1).

Recall that the solutions of $y' + cy = 0$ is $y = Ke^{-cy}$ where K is a constant. This suggests that $y = e^{rt}$ for certain constant r may be a solution of the above homogeneous second-order differential equations. Substituting $y = e^{rt}$ into (4.2.1), we get $e^{rt}(ar^2 + br + c) = 0$. Thus if r is a solution of the **characteristic equation**

$$ar^2 + br + c = 0 \tag{4.2.2}$$

then $y = e^{rt}$ is a solution of the homogeneous differential equation (4.2.1). The roots of (4.2.2) are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are 3 cases:

1. $b^2 - 4ac > 0$. Then r_1 and r_2 are distinct roots of (4.2.2). So both $y = e^{r_1 t}$ and $y = e^{r_2 t}$ are solutions of the homogeneous equation (4.2.1). The general solutions of (4.2.1) are:
$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \tag{4.2.3a}$$
where C_1 and C_2 are two constants.

2. $b^2 - 4ac = 0$. Then (4.2.2) has a repeated root r. The general solutions of (4.2.1) are:
$$y = C_1 e^{rt} + C_2 t e^{rt} \tag{4.2.3b}$$

3. $b^2 - 4ac < 0$. Then (4.2.2) has two complex conjugated roots
$$r_1 = p + q\sqrt{-1} \qquad \text{and} \qquad r_2 = p - q\sqrt{-1}.$$
The general solutions of (4.2.1) are:
$$y = C_1 e^{pt} \sin(qt) + C_2 e^{pt} \cos(qt). \tag{4.2.3c}$$
The general solutions are essentially the same as (4.2.3a). This will be discuss in Section 4.3.

Ex. (ex1, p170) Find solutions of $y'' + 5y' - 6y = 0$.

Ex. (ex2, p170) Solve the IVP
$$y'' + 2y' - y = 0; \qquad y(0) = 0, \quad y'(0) = -1.$$

Theorem 4.1. *For any real numbers a ($\neq 0$), b, c, t_0, Y_0, and Y_1, there exists a unique solution for all $t \in (-\infty, \infty)$ to the IVP*
$$ay'' + by' + cy = 0; \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1. \tag{4.2.4}$$

We notice that the general solutions of $ay'' + by' + cy = 0$ are always expressed as a linear combination of two different functions. This suggests us to define "different".

Def 4.2. Two functions $y_1(t)$ and $y_2(t)$ are **linearly independent on the interval** I if and only if neither of them is a constant multiple of the other on I. Otherwise, we say that they are **linearly dependent on** I.

The linear dependence of two functions can be checked by Wronskian:

Theorem 4.3. *Two smooth functions $y_1(t)$ and $y_2(t)$ are linearly dependent on I if and only if their* **Wronskian** *is always zero on I:*

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) = 0 \quad \textit{for all} \quad t \in I. \tag{4.2.5}$$

Ex. (ex3, p174) Solve the IVP: $\quad y'' + 4y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 3.$

In general, we can solve the homogeneous constant coefficient linear n-th order differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \tag{4.2.6}$$

by solving the characteristic equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_n = 0.$$

Suppose it has distinct roots $r_1, r_2, \cdots, r_m$, where r_i is repeated ℓ_i times, then the general solutions of (4.2.6) are the linear combinations of

$$e^{r_i t}, \quad te^{r_i t}, \quad t^2 e^{r_i t}, \quad \cdots, \quad t^{\ell_i - 1} e^{r_i t}$$

for $i = 1, 2, \cdots, m$.

Ex. (ex4, p175) Find the general solutions of $y''' + 3y'' - y' - 3y = 0$.

Homework

4.2 (p176-178): 1, 3, 5, 13, 19, 27, 28, 29, 39

4.3 Auxiliary Equations with Complex Roots

If the auxiliary equation $ar^2 + br + c = 0$ has two distinct complex conjugate roots $r_1 = p + \mathbf{i}q$ and $r_2 = p - \mathbf{i}q$ where $\mathbf{i} = \sqrt{-1}$, then the solutions of $ay'' + by' + cy = 0$ are

$$y(t) = C_1 e^{(p+\mathbf{i}q)t} + C_2 e^{(p-\mathbf{i}q)t} = C_1 e^{pt} e^{\mathbf{i}qt} + C_2 e^{pt} e^{-\mathbf{i}qt}. \tag{4.3.1}$$

What are $e^{\mathbf{i}qt}$ and $e^{-\mathbf{i}qt}$?

By Taylor expansion,

$$\begin{aligned}
e^{\mathbf{i}\theta} &= 1 + (\mathbf{i}\theta) + \frac{(\mathbf{i}\theta)^2}{2!} + \cdots + \frac{(\mathbf{i}\theta)^n}{n!} + \cdots \\
&= 1 + \mathbf{i}\theta - \frac{\theta^2}{2!} - \frac{\mathbf{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\mathbf{i}\theta^5}{5!} - \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + \mathbf{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + \mathbf{i}\sin\theta.
\end{aligned}$$

It is called **Euler's formula**: $e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$.

Now the expression (4.3.1) becomes

$$\begin{aligned}
y(t) &= C_1 e^{pt} e^{\mathbf{i}qt} + C_2 e^{pt} e^{-\mathbf{i}qt} \\
&= C_1 e^{pt}(\cos(qt) + \mathbf{i}\sin(qt)) + C_2 e^{pt}(\cos(-qt) + \mathbf{i}\sin(-qt)) \\
&= (C_1 + C_2)e^{pt}\cos(qt) + (\mathbf{i}C_1 + \mathbf{i}C_2)e^{pt}\sin(qt) \\
&= B_1 e^{pt}\cos(qt) + B_2 e^{pt}\sin(qt)
\end{aligned}$$

Ex. Solve $y'' + y = 0$.

Ex. (ex1, p180) Solve the IVP

$$y'' + 2y' + 2y = 0; \qquad y(0) = 0, \quad y'(0) = 2.$$

Ex. (ex3, p180) The mechanics of the mass-spring oscillator is governed by $my''(t) + by'(t) + ky(t) = 0$ ($y(t)$ is the displacement function), where m =inertia, b =damping, k =stiffness.

Solve the equation of motion when $m = 36\ kg$, $b = 12\ kg/sec$, $k = 37\ kg/sec^2$, $y(0) = 0.7\ m$, and $y'(0) = 0.1\ m/sec$. Also find $y(10)$, the displacement after 10 *sec*.

Ex. (ex4, p183) Interpret the equation $36y'' - 12y' + 37y = 0$ in terms of the mass-spring system. (Fig 4.7)

Homework

4.3 (p186-188): 1, 3, 9, 11, 21, 23, 26, 28, 32, 33

4.4 Nonhomogeneous Equations

We turn to **nonhomogeneous** linear equation with constant coefficients

$$ay'' + by' + cy = f(t). \tag{4.4.1}$$

If $y_1(t)$ is a particular solution of (4.4.1), then the general solution of (4.4.1) is $y(t) = y_1(t) + y_2(t)$, where $y_2(t)$ is a general solution of $ay'' + by' + cy = 0$. Thus it suffices to find out *just one* solution of (4.4.1).

In this section, we will find a particular solution to the nonhomogeneous equation of the form

$$ay'' + by' + cy = Ct^m e^{rt}. \tag{4.4.2}$$

We guess that $y_p(t) = h(t)e^{rt}$ for some polynomial $h(t)$ may be a solution of (4.4.2). The key is to determine the polynomial $h(t)$ using method of undetermined coefficients.

Likewise, for the nonhomogeneous equation of the form

$$\begin{aligned} ay'' + by' + cy &= Ct^m e^{\alpha t}\cos\beta t \qquad \text{or} \\ ay'' + by' + cy &= Ct^m e^{\alpha t}\sin\beta t \end{aligned} \tag{4.4.3}$$

we guess that $y_p(t) = h_1(t)e^{\alpha t}\cos\beta t + h_2(t)e^{\alpha t}\sin\beta t$ may be a solution of (4.4.3), where $h_1(t)$ and $h_2(t)$ are certain polynomials. Again, we should determine the polynomials $h_1(t)$ and $h_2(t)$.

Ex. (ex1, p188) $y'' + 3y' + 2y = 3t$. (Try $y_p(t) = At + B$)

In general, to solve $ay'' + by' + cy = f(t)$ where $f(t)$ is a polynomial of degree m, we may try $y_p(t) = h(t)$ where $h(t)$ is a polynomial of degree m with coefficients to be determined.

Ex. (ex2, p189) $y'' + 3y' + 2y = 10e^{3t}$. (Try $y_p(t) = Ae^{3t}$)

Ex. (ex3, p189) $y'' + 3y' + 2y = \sin t$. (Try $y_p(t) = A\sin t + B\cos t$)

Ex. (ex4, p190) $y'' + 4y = 5t^2e^t$. (Try $y_p(t) = (At^2 + Bt + C)e^t$)

A tricky thing: The solutions of $ay'' + by' + cy = Ct^m e^{rt}$ vary when r is (*not a root*/*a simple root*/*a double root*) of the associated auxiliary equation $ax^2 + bx + c = 0$. We state the following theorem and verify it by examples.

Theorem 4.4 (Method of Undetermined Coefficients). *To find a solution to* $ay'' + by' + cy = Ct^m e^{rt}$*, use the form*

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt} \tag{4.4.4}$$

with $s \in \{0, 1, 2\}$ *equals to the multiplicity of* $(x - r)$ *in* $(ax^2 + bx + c)$*:*

$$\begin{cases} s = 0 & \text{if } r \text{ is } \underline{\text{not a root}} \text{ of the assoc. aux. equation.} \\ s = 1 & \text{if } r \text{ is } \underline{\text{a simple root}} \text{ of the assoc. aux. equation.} \\ s = 2 & \text{if } r \text{ is } \underline{\text{a double root}} \text{ of the assoc. aux. equation.} \end{cases}$$

To find a solution of $ay'' + by' + cy = t^m e^{\alpha t}(M \cos \beta t + N \sin \beta t)$*, use the form*

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0) e^{\alpha t} \cos \beta t + t^s(B_m t^m + \cdots + B_1 t + B_0) e^{\alpha t} \sin \beta t$$

with $s \in \{0, 1\}$ *equals to the multiplicity of* $[x - (\alpha + \mathbf{i}\beta)]$ *in* $(ax^2 + bx + c)$*:*

$$\begin{cases} s = 0 & \text{if } \alpha + \mathbf{i}\beta \text{ is } \underline{\text{not a root}} \text{ of the assoc. aux. equation} \\ s = 1 & \text{if } \alpha + \mathbf{i}\beta \text{ is } \underline{\text{a root}} \text{ of the assoc. aux. equation} \end{cases}$$

Ex. (ex5, p193) Find the form of a solution to $y'' + 2y' - 3y = f(t)$ where $f(t)$ equals
(a) $7 \cos 3t$ (b) $2te^t \sin t$ (c) $t^2 \cos \pi t$ (d) $5e^{-3t}$ (e) $3te^t$ (f) $t^2 e^t$

Ex. (ex6, p194) Find the form of a solution to $y'' - 2y' + y = f(t)$ where $f(t)$ equals
(a) $7 \cos 3t$ (b) $2te^t \sin t$ (c) $t^2 \cos \pi t$ (d) $5e^{-3t}$ (e) $3te^t$ (f) $t^2 e^t$

Ex. (ex7, p194) Find the form of a solution to $y'' - 2y' + 2y = 5te^t \cos t$.

Homework

4.4 (p195): 1-8, 10-20, 27-32

4.5 The Superposition Principle and Undetermined Coefficients Revisited

Theorem 4.5 (Superposition Principle). *Let* y_1 *be a solution to* $ay'' + by' + cy = f_1(t)$ *and let Let* y_2 *be a solution to* $ay'' + by' + cy = f_2(t)$*. Then for any constants* c_1 *and* c_2*, the function* $c_1 y_1 + c_2 y_2$ *is a solution to* $ay'' + by' + cy = c_1 f_1(t) + c_2 f_2(t)$*.*

Ex. (ex1, p196) Find a solution to
(1) $y'' + 3y' + 2y = 3t + 10e^{3t}$ and (2) $y'' + 3y' + 2y = -9t + 20e^{3t}$.

(*hint: We found that* $y_1(t) = \frac{3t}{2} - \frac{9}{4}$ *solved* $y'' + 3y' + 2y = 3t$*, and* $y_2(t) = e^{3t}/2$ *solved* $y'' + 3y' + 2y = 10e^{3t}$.)

Theorem 4.6 (Existence and Uniqueness: Nonhomogeneous Case). *Suppose* $y_p(t)$ *is a particular solution to* $ay'' + by' + cy = f(t)$ *in an interval* I *containing* t_0 *and that* $y_1(t)$ *and* $y_2(t)$ *are linearly independent solutions to the associated homogeneous equation* $ay'' + by' + cy = 0$*. Then there exists a unique solution in* I *to the initial value problem*

$$ay'' + by' + cy = f(t), \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1,$$

for any a, b, c, t_0, Y_0, Y_1*, and it is given by* $y(t) = y_p(t) + c_1 y_1(t) + c_2 y_2(t)$ *for some constants* c_1 *and* c_2.

Ex. (ex2, p198) Given that $y_p(t) = t^2$ is a particular solution to $y'' - y = 2 - t^2$, find a general solution and a solution of the IVP $y(0) = 1$, $y'(0) = 0$.

Ex. (ex3, p198) A mass-spring system is governed by $y'' + 2y' + 2y = 5\sin t + 5\cos t$. If the mass is initially located at $y(0) = 1$, with a velocity $y'(0) = 2$, find its equation of motion.

Ex. (ex4, p199) Find a solution to $y'' - y = 8te^t + 2e^t$.

Ex. (ex5, p200) Write down the form of a solution to $y'' + 2y' + 2y = 5e^{-t}\sin t + 5t^3e^{-t}\cos t$.

Ex. (ex6, p200) Write down the form of a solution to $y''' + 2y'' + y' = 5e^{-t}\sin t + 3 + 7te^{-t}$.

Homework

4.5 (p201-203): 1, 2, 3, 7, 17, 20, 25, 26, 31-36

4.6 Variation of Parameters

Variation of parameters is a more general method to solve a particular solution of $ay'' + by' + cy = g(t)$.

1. Solve the homogeneous equation

$$ay'' + by' + cy = 0 \overset{\text{general solution}}{\Longrightarrow} y_h(t) = c_1 y_1(t) + c_2 y_2(t). \qquad (4.6.1)$$

2. For the nonhomogeneous equation, we variate the constants c_1 and c_2 to seek a particular solution of the form:

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \qquad (4.6.2)$$

3. Compute

$$y_p' = (v_1'y_1 + v_2'y_2) + (v_1y_1' + v_2y_2')$$

   To simplify the expression, we assume that

$$v_1'y_1 + v_2'y_2 = 0. \qquad (4.6.3)$$

   So $y_p' = v_1y_1' + v_2y_2'$.

4. Now $y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$. Substitute y_p, y_p', and y_p'' into the nonhomogeneous equation. We get

$$v_1'y_1' + v_2'y_2' = \frac{g}{a} \qquad (4.6.4)$$

5. Solve the linear system (4.6.3) and (4.6.4) to get v_1' and v_2', to get v_1 and v_2, and then get $y_p(t)$.

Theorem 4.7 (Method of Variation of Parameters). *To determine a solution to $ay'' + by' + cy = g$;*

1. *Find two linearly independent solutions $\{y_1(t), y_2(t)\}$ of the corresponding homogeneous equation and take*

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t).$$

2. *Determine $v_1(t)$ and $v_2(t)$ by solving v_1' and v_2' and integrating:*

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = \frac{g}{a} \end{cases}$$

Ex. (ex1, p205) Find a general solution on $(-\pi/2, \pi/2)$ to $\dfrac{d^2y}{dt^2} + y = \tan t$.

Ex. (ex2, p206) Find a particular solution on $(-\pi/2, \pi/2)$ to

$$\frac{d^2y}{dt^2} + y = \tan t + 3t - 1.$$

Homework

4.6 (p206-207): 1, 2, 4, 5, 10, 12, 16, 18

4.7 Variable Coefficient Equations

This section is not required. However, the basic idea is similar to that for constant coefficient equations.

4.8 Vibration Problems

We look in details of vibration problems. It covers some materials in Section 4.9 (Free Mechanical Vibrations) and Section 4.8 (Qualitative Considerations for Variable-Coefficient and Nonlinear Equations). Recall that the governing equation of a mass-spring system is

$$\begin{aligned} F_{ext} &= [\text{inertia}]\frac{d^2y}{dt^2} + [\text{damping}]\frac{dy}{dt} + [\text{stiffness}]y \\ &= my'' + by' + ky \end{aligned}$$

4.8.1 Free Mechanical Vibrations

1. The **undamped, free system** is $b = 0$ and $F_{ext} = 0$, so that

$$m\frac{d^2y}{dt^2} + ky = 0.$$

Set $\omega := \sqrt{k/m}$. Then $y'' + \omega^2 y = 0$ and the general solution is

$$\begin{aligned} y(t) &= C_1 \cos\omega t + C_2 \sin\omega t \\ &= A\sin(\omega t + \phi) \end{aligned}$$

where $A = \sqrt{C_1^2 + C_2^2}$ and $\tan\phi = \dfrac{C_1}{C_2}$.

Conclusion: The motion of a mass in an undamped, free system is a sine wave, with **period** $2\pi/\omega$ (in *sec*) and **natural frequency** $\omega/2\pi$ (in *cycles/sec*), where the **angular frequency** $\omega = \sqrt{k/m}$ (in *rad/sec*).

Ex. (ex1, p231)

2. Now suppose the mass-spring system has no external force but there exists **damping force** affecting the vibration. The equation is

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

The roots of the characteristic equation $mr^2 + br + k = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4mk} \tag{4.8.1}$$

The forms of the solutions depend on the discriminant $b^2 - 4mk$.

(a) ($b^2 < 4mk$) Underdamped or Oscillatory Motion.
The characteristic equation has two complex roots $\alpha \pm \mathbf{i}\beta$ where

$$\alpha = -\frac{b}{2m}, \qquad \beta = \frac{1}{2m}\sqrt{4mk - b^2}$$

A general solution of $my'' + by' + ky = 0$ is

$$y(t) = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t) = Ae^{\alpha t}\sin(\beta t + \phi)$$

where $A = \sqrt{C_1^2 + C_2^2}$ and $\tan\phi = C_1/C_2$. (See Fig 4.28 at p233)

(b) ($b^2 > 4mk$) Overdamped Motion.
The characteristic equation has two distinct real roots

$$r_1 = -\frac{b}{2m} + \frac{1}{2m}\sqrt{b^2 - 4mk}, \qquad r_2 = -\frac{b}{2m} - \frac{1}{2m}\sqrt{b^2 - 4mk}.$$

A general solution is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Since m, b, k are positive, both r_1 and r_2 are negative. So $y(t) \to 0$ as $t \to \infty$. Moreover,

$$y'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t} = e^{r_1 t}(C_1 r_2 + C_2 r_2 e^{(r_2 - r_1)t})$$

either $y'(t) \equiv 0$ (in which $y(t) \equiv 0$), or $y'(t) = 0$ for AT MOST ONE t. So the overdamped motion does not oscillate. (See Fig 4.29 on p234)

(c) ($b^2 = 4mk$) Critically Damped Motion.
The characteristic equation has repeated roots $-b/2m$. A general solution to $my'' + by' + ky = 0$ is

$$y(t) = (C_1 + C_2 t)e^{-(b/2m)t}$$

First by L'Hôpital's rule, $y(t) \to 0$ as $t \to \infty$. Next,

$$y'(t) = (C_2 - \frac{b}{2m}C_1 - \frac{b}{2m}C_2 t)e^{-(b/2m)t}$$

So either $y'(t) \equiv 0$ (where $y(t) \equiv 0$), or $y'(t) = 0$ for AT MOST ONE t. Critically damped motions are similar to overdamped motions.
Ex. (ex2, p235) [briefly discuss three situations].

Ex. (ex3, p236)

4.8.2 Qualitative Consideration by Mass-Spring Model

1. A special type of equation $y'' = f(y)$ can be solved by multiplying y' on both sides:

$$\begin{aligned} y'y'' = y'f(y) &\implies \int y'y''dt = \int y'f(y)dt \\ &\implies \frac{1}{2}(y')^2 = \int f(y)dy + K := F(y) + K \\ &\implies y' = \pm\sqrt{2[F(y)+K]} \\ &\implies t = \pm\int \frac{dy}{\sqrt{2[F(y)+K]}} + C. \end{aligned}$$

Ex. (ex1, p220) $y'' = 6y^2$. (Note: The performance is totally different from linear equations. For examples, the solution can blow up anywhere, we have infinite many linearly independent solutions, etc.)

2. For a general variable-coefficient or non-linear differential equation

$$m(y,t)\frac{d^2y}{dt^2} + b(y,t)\frac{dy}{dt} + k(y,t)y = g(y,t)$$

We may use mass-spring model to describe some performance of the solution curve.

Ex. (ex3, p221) Using the mass-spring analogy, predict the nature of the solutions to $y'' + ty = 0$ for $t > 0$.

Ex. (ex4, p222) Apply the mass-spring analogy to predict qualitative features of solutions to Bessel's equation $y'' + \frac{1}{t}y' + (1 - \frac{n^2}{t^2})y = 0$ for $t > 0$.

Ex. (ex6, p224) Predict the qualitative features of the solutions to the nonlinear Duffing equation $y'' + y + y^3 = 0$.

Ex. (ex7, p225) Predict the behavior of the solutions to van der Pol equation $y'' - (1 - y^2)y' + y = 0$.

Homework

4.8-4.9 (p227-229, p238-239):
Section 4.8: 1, 4, 5, 11, 15
Section 4.9: 1, 3, 5, 6, 11, 16