Chapter 6

Theory of Higher-Order Linear Differential Equations

6.1 Basic Theory

A **linear differential equation of order** n has the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) = b(x), \tag{6.1.1}$$

where $a_0(x), \cdots, a_n(x)$ and $b(x)$ depend only on x.

constant coefficients: When $a_0, a_1, \cdots, a_n$ are all constants.
variable coefficients: When some of $a_0, a_1, \cdots, a_n$ are not constants.
homogeneous: When $b(x) = 0$.
nonhomogeneous: When $b(x) \neq 0$.
standard form: Divide (6.1.1) by $a_n(x)$ to get

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)$$

Theorem 6.1 (Existence and Uniqueness). *If $p_1(x), \cdots, p_n(x)$ and $g(x)$ are each continuous on (a, b) that contains x_0. Then for any initial values $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$, there exists a unique solution $y(x)$ on the whole interval (a, b) to the following IVP:*

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x), \tag{6.1.2}$$

$$y(x_0) = \gamma_0, \quad y(x_1) = \gamma_1, \quad \cdots, \quad y(x_{n-1}) = \gamma_{n-1}. \tag{6.1.3}$$

Ex. (ex1, p343)

Def 6.2. Let $f_1, \cdots, f_n$ be any n functions that are $(n-1)$ times differentiable. The function

$$W[f_1, \cdots, f_n] = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of $f_1, \cdots, f_n$. (What is determinant?)

Theorem 6.3 (Solutions for Homogeneous Case). *Given n solutions $y_1, \cdots, y_n$ to*

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$$

where $p_1, \cdots, p_n$ are continuous on (a, b). If for some points $x_0 \in (a, b)$ there is

$$W[y_1, \cdots, y_n](x_0) \neq 0,$$

then every solution of the above homogeneous equation on (a, b) can be expressed by

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x).$$

n functions $f_1, \cdots, f_n$ are **linearly dependent on an interval** I if one of them can be expressed as a linear combination of the others. Equivalently, if there exists constants $c_1, \cdots, c_n$ such that

$$c_1 f_1(x) + \cdots + c_n f_n(x) \equiv 0$$

for all x on I. Otherwise, they are **linearly independent on** I.

Ex. (ex2, p346)

Ex. (ex3, p346)

For solutions $y_1, \cdots, y_n$ of the above homogeneous equation, the linear dependence can be checked by its Wronskian at on point.

Theorem 6.4. *If $y_1, \cdots, y_n$ are solutions to $y^{(n)}(x)+p_1(x)y^{(n-1)}(x)+\cdots+p_n(x)y(x) = 0$ on (a, b) with $p_1, \cdots, p_n$ continuous on (a, b), then $y_1, \cdots, y_n$ are linearly dependent on (a, b) if and only if $W[y_1, \cdots, y_n](x_0) = 0$ for some x_0 on (a, b).*

Theorem 6.5 (Solutions for Nonhomogeneous Case). *If $y_p(x)$ is a particular solution to the nonhomogeneous equation*

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x) \tag{6.1.4}$$

on (a, b) with $p_1, \cdots, p_n$ continuous on (a, b), and $\{y_1, \cdots, y_n\}$ is a fundamental solution set of the corresponding homogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0$$

then the general solution of (6.1.4) *is of the form*

$$y(x) = y_p(x) + C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x).$$

Ex. (ex4, p349)

Homework

6.1 (p349-351): 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23

6.2 Homogeneous Linear Equations with Constant Coefficients

The solutions to a homogeneous linear equation with constant coefficients

$$a_ny^{(n)} + \cdots + a_1y' + a_0y = 0 \tag{6.2.1}$$

depends on the roots of the **characteristic equation**

$$P(r) := a_nr^n + \cdots + a_1r + a_0 = 0 \tag{6.2.2}$$

Let D be the operator such that $D(y) = y'$. Then $P(D) = a_nD^n + \cdots + a_1D + a_0$ is an operator. The homogeneous equation (6.2.1) becomes $P(D)(y) = 0$. If $P(r)$ has distinct roots $r_1, \cdots, r_m$ where r_i repeats d_i times ($d_i = 1, 2, \cdots$, $d_1 + \cdots + d_m = n$), then

$$P(D)(y) = (D - r_1)^{d_1} \cdots (D - r_m)^{d_m}(y) = 0$$

Clearly, if y is a solution to $(D - r_i)^{d_i}(y) = 0$ for some i, then y is a solution to $P(D)(y) = 0$.

One can verify by induction that: A set of fundamental solutions to the differential equation $(D - r_i)^{d_i}(y) = 0$ is $\{e^{r_it}, te^{r_it}, \cdots, t^{d_i-1}e^{r_it}\}$. So $P(D)(y) = 0$ has the following solutions:

Theorem 6.6. *A fundamental solution set* $\{y_1, \cdots, y_n\}$ *of the homogeneous constant coefficient equation* $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$ *consists of*

$$e^{r_i t}, te^{r_i t}, \cdots, t^{d_i - 1} e^{r_i t}$$

where r_i *is a root repeated* d_i *times* ($d_i = 1, 2, \cdots$) *in the characteristic equation* $a_n r^n + \cdots + a_1 r + a_0 = 0$.

Ex. (ex1, p353, distinct roots)

Ex. (ex3, p356, repeated roots)

If the characteristic equation $P(r) = 0$ has complex conjugated roots $\alpha + \mathbf{i}\beta$ and $\alpha - \mathbf{i}\beta$, *each repeated* d *times*, then the solution functions

$$\{e^{(\alpha+\mathbf{i}\beta)t}, \ \cdots, t^{d-1} e^{(\alpha+\mathbf{i}\beta)t}, \ \ e^{(\alpha-\mathbf{i}\beta)t}, \ \cdots, t^{d-1} e^{(\alpha-\mathbf{i}\beta)t}\}$$

can be replaced by real functions

$$\{e^{\alpha t}\cos(\beta t), \ \cdots, \ t^{d-1} e^{\alpha t}\cos(\beta t), \ \ e^{\alpha t}\sin(\beta t), \ \cdots, \ t^{d-1} e^{\alpha t}\sin(\beta t)\}.$$

Ex. (ex2, p354)

Ex. (ex4, p356)

Ex. (hw18, p356)

Homework

6.2 (p356-358): 1, 2, 3, 9, 11, 14, 15, 17, 19, 21

6.3 Undetermined Coefficients and the Annihilator Method

We have used undetermined coefficient method to solve the linear differential equation $(aD^2 + bD + c)[y] = f(x)$, where $f(x)$ is of the form $Ct^m e^{rt}$, $Ct^m e^{\alpha t}\cos(\beta t)$, or $Ct^m e^{\alpha t}\sin(\beta t)$, or a summand of such terms by superposition principle.

The method of undetermined coefficients can be generalized to solve $P(D)[y] = f(x)$ where $f(x)$ is of the above forms. This can be justified by the annihilator method below.

To solve $P(D)[y] = f(x)$, suppose $f(x)$ is a solution to $Q(D)[f] = 0$ for some polynomial $Q(r)$ (so $Q(r)$ is said to **annihilate** f), then

$$Q(D)P(D)[y] = Q(D)[f](x) = 0.$$

In other words,

$$\boxed{y \text{ is a solution to } P(D)[y] = f(x) \implies y \text{ is a solution to } Q(D)P(D)[y] = 0.}$$

The latter one is homogeneous and we know how to solve it. Thus we may choose y from a general solution of $Q(D)P(D)[y] = 0$, then apply the method of undetermined coefficients to get the coefficients.

Lemma 6.7. The following rules are useful to determine annihilators of certain functions:

1. $f(x) = e^{rx}$ satisfies $(D - r)[f] = 0$. (i.e. $(D - r)$ annihilates e^{rx})
2. $f(x) = x^k e^{rx}$ satisfies $(D - r)^m[f] = 0$ for $k = 0, 1, \cdots, m - 1$.
3. $f(x) = \cos\beta x$ or $\sin\beta x$ satisfies $(D^2 + \beta^2)[f] = 0$.
4. $f(x) = x^k e^{\alpha x}\cos(\beta x)$ or $x^k e^{\alpha x}\sin(\beta x)$ satisfies $[(D-\alpha)^2 + \beta^2]^m[f] = 0$ for $k = 0, 1, \cdots, m - 1$.
5. If $Q_i(D)$ is an annihilator of $f_i(x)$ for $i = 1, \cdots, p$, then the least common multipler $\text{lcm}(Q_1(D), \cdots, Q_p(D))$ and the product $Q_1(D)\cdots Q_p(D)$ are both annihilators of $f_1(x) + f_2(x) + \cdots + f_p(x)$.

(proof)

Ex. (ex1, p359) Find a differential operator that annihilates $6xe^{-4x} + 5e^x\sin(2x)$.

Ex. (ex2, p360) Use both method of undetermined coefficient and method of annihilators to find a general solution to $y'' - y = xe^x + \sin x$.

Ex. (ex3, p361) Use the annihilator method to find a general solution to $y''' - 3y'' + 4y = xe^{2x}$.

Theorem 6.8 (Method of Undetermined Coefficients). *Let $L = P(D)$ be a constant coefficient polynomial differential operator.*

1. *To find a particular solution to $L[y] = Cx^m e^{rx}$, use the form*

$$y_p(x) = x^s[A_m x^m + \cdots + A_1 x + A_0]e^{rx}$$

where $s \in \{0, 1, 2, \cdots\}$ is the multiplicity of r in the associated characteristic equation $P(z) = 0$.

2. *To find a particular solution to $L[y] = Cx^m e^{\alpha x}\cos(\beta x)$ or $L[y] = Cx^m e^{\alpha x}\sin(\beta x)$, use the form*

$$y_p(x) = x^s[A_m x^m + \cdots + A_1 x + A_0]e^{\alpha x}\cos(\beta x) + x^s[B_m x^m + \cdots + B_1 x + B_0]e^{\alpha x}\sin(\beta x)$$

where $s \in \{0, 1, 2, \cdots\}$ is the multiplicity of $\alpha + \mathbf{i}\beta$ in the associated characteristic equation $P(z) = 0$.

Homework

6.3 (p362-363): 1, 3, 5, 7, 9, 11, 13, 15, 23, 25, 27, 31, 33