Chapter 7

Laplace Transforms

7.1 Introduction: A Mixing Problem

7.2 Definition of the Laplace Transform

Def 7.1. Let $f(t)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function $F(s)$ defined by the integral

$$F(s) := \int_0^\infty e^{-st} f(t)dt \qquad \text{(an improper integral)}$$

The Laplace transform of f is defined by both F and $\mathcal{L}\{f\}$. The domain of F is all s where the above integral converges.

Ex. (ex1, p378) Determine the Laplace transform for $f(t) = 1$, $t \geq 0$.

Ex. (ex2, p378) Determine the Laplace transform for $f(t) = e^{\alpha t}$.

Ex. (ex3, p379) Find $\mathcal{L}\{\sin bt\}$. (Briefly go through it in the textbook)

Ex. (ex4, p378) Determine the Laplace transform of $f(t) = \begin{cases} 2, & 0 < t < 5 \\ 0, & 5 < t < 10 \\ e^{4t}, & 10 < t \end{cases}$

Theorem 7.2 (Linearity). *Let f, f_1, and f_2 be functions whose Laplace transforms exist and let c be a constant. Then for s in the common domain of these Laplace transforms,*

$$\begin{aligned} \mathcal{L}\{f_1 + f_2\} &= \mathcal{L}\{f_1\} + \mathcal{L}\{f_2\}, \\ \mathcal{L}\{cf\} &= c\mathcal{L}\{f\}. \end{aligned}$$

Ex. (ex5, p380) Determine $\mathcal{L}\{11 + 5e^{4t} - 6\sin 2t\}$.

The Laplace transform exist for all integrable functions on $[0, \infty)$ that "not going too fast to ∞".

A function $f(t)$ is said to be **piecewise continuous on** $[0, \infty)$ if $f(t)$ is continuous at all but finitely many points in $[0, N]$ for all $N > 0$.

$f(t)$ is said to be of **exponential order** α if there exists $T > 0$ and $M > 0$ such that

$$|f(t)| \leq Me^{\alpha t}, \qquad \text{for all} \ \ t \geq T.$$

Theorem 7.3. *If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential α, then $\mathcal{L}\{f\}(s)$ exists for all $s > \alpha$.*

We have a Table of Important Laplace Transforms:

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	Domain of $F(s)$
1	$\frac{1}{s}$,	$s > 0$
e^{at}	$\frac{1}{s-a}$,	$s > a$
t^n	$\frac{n!}{s^{n+1}}$,	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$,	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$,	$s > 0$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$,	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$,	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$,	$s > a$

Homework

7.2 (p385-386): 3, 4, 5, 8, 9, 13, 17, 19

7.3 Properties of the Laplace Transform

We defined the Laplace transform $\mathcal{L}\{f\}(s) = \displaystyle\int_0^\infty e^{-st} f(t) ds$ in last section, and show that it satisfies the linearity. There are more nice properties for the Laplace transforms.

Theorem 7.4. *Suppose the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then*

1. *(Translation in s)* $\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$ *for* $s > \alpha + a$.

2. *(Derivative) If f' exists, then for $s > \alpha$,*

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

3. *(Higher-Order Derivatives) If $f, f', \cdots, f^{(n)}$ exist and piecewise continuous on $[0, \infty)$, then for $s > \alpha$,*

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

4. *(Derivatives of the Laplace Transform) Let $F(s) = \mathcal{L}\{f\}(s)$. Then for $s > \alpha$,*

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{d s^n}(s)$$

(Proofs of 1, 2, 4)

Ex. (ex1, p387) $\mathcal{L}\{e^{at}\sin bt\}$.

Ex. (ex2, p388) Use $\mathcal{L}\{\sin bt\}(s) = \dfrac{b}{s^2+b^2}$ to determine $\mathcal{L}\{\cos bt\}$.

Ex. (ex3, p388) Prove that for continuous function $f(t)$,

$$\mathcal{L}\{\int_0^t f(\tau)\, d\tau\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s).$$

Ex. (ex4, p390) $\mathcal{L}\{t\sin bt\}$.

Table 7.2 summarizes these basic properties of the Laplace transform.

Homework

7.3 (p391-392): 1, 3, 7, 9, 13, 21, 22, 24, 25

7.4 Inverse Laplace Transform

7.4.1 Definition

Ex. To solve $y'' - y = -t; \quad y(0) = 0, \quad y'(0) = 1$, we first apply Laplace transform $Y(s) = \mathcal{L}\{y\}(s)$ to get

$$\mathcal{L}\{y''\}(s) - Y(s) = -\frac{1}{s^2}.$$

Notice that $\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1$. So

$$s^2Y(s) - 1 - Y(s) = -\frac{1}{s^2}.$$

We solve that $Y(s) = \mathcal{L}\{y\}(s) = \dfrac{1}{s^2}$. Since $\mathcal{L}\{t\}(s) = \frac{1}{s^2}$, it is reasonable to conclude that $y(t) = t$ is the solution to the original initial value problem.

Def 7.5. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies $\mathcal{L}\{f\} = F$, then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notion $f = \mathcal{L}^{-1}\{F\}$.

Table 7.1 and Table 7.2 are useful to compute both Laplace transforms and inverse Laplace transforms.

Ex. (ex1, p393) Determine $\mathcal{L}^{-1}\{F\}$, where
(a) $F(s) = \dfrac{2}{s^3}$, (b) $F(s) = \dfrac{3}{s^2+9}$, (c) $F(s) = \dfrac{s-1}{s^2-2s+5}$.

Theorem 7.6 (Linearity). *Suppose $\mathcal{L}^{-1}\{F\}$, $\mathcal{L}^{-1}\{F_1\}$, and $\mathcal{L}^{-1}\{F_2\}$ are continuous on $[0, \infty)$. Then*

1. $\mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}$.
2. $\mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}$.

The other properties of Laplace transform can be transferred into those for inverse Laplace transform.

7.4.2 Inverse Laplace Transforms of Rational Functions

Ex. (ex2, p394) $\mathcal{L}^{-1}\{\dfrac{5}{s-6} - \dfrac{6s}{s^2+9} + \dfrac{3}{2s^2+8s+10}\}$.

Ex. (ex3, p395) $\mathcal{L}^{-1}\{\frac{5}{(s+2)^4}\}$.

Ex. (ex4, p395) $\mathcal{L}^{-1}\{\frac{3s+2}{s^2+2s+10}\}$.

To compute the inverse Laplace transform of a real coefficient rational function $\frac{P(s)}{Q(s)}$, we may use the **method of partial fraction**:

1. Decompose the denominator $Q(s)$ into **irreducible** linear factors $(s-r)$ and $[(s-\alpha)^2+\beta^2]$. Suppose

$$Q(s) = C \cdot \prod_{i=1}^{p}(s-r_i)^{n_i} \cdot \prod_{j=1}^{q}[(s-\alpha_j)^2+\beta_j^2]^{m_j}$$

2. Then $\frac{P(s)}{Q(s)}$ is a summand of the partial fractions

$$\frac{A_1}{s-r_i} + \frac{A_2}{(s-r_i)^2} + \cdots + \frac{A_{n_i}}{(s-r_i)^{n_i}}$$

   and

$$\frac{B_1 s + C_1}{(s-\alpha_j)^2+\beta_j^2} + \frac{B_2 s + C_2}{[(s-\alpha_j)^2+\beta_j^2]^2} + \cdots + \frac{B_{m_j} s + C_{m_j}}{[(s-\alpha_j)^2+\beta_j^2]^{m_j}}.$$

3. We first determine the coefficients A_i, B_j, C_k, then compute the inverse Laplace transform of each term.

1. Nonrepeated Linear Factors

   Ex. (ex5, p396) Determine $\mathcal{L}^{-1}\{F\}$, where $F(s) = \frac{7s-1}{(s+1)(s+2)(s+3)}$.

2. Repeated Linear Factors

   Ex. (ex6, p398) Determine $\mathcal{L}^{-1}\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\}$.

3. Quadratic Factors

   Ex. (ex7, p399) Determine $\mathcal{L}^{-1}\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\}$.

The situation of repeated quadratic factors will be discussed later.

Homework

7.4 (p400-402): 1, 2, 3, 9, 11, 13, 15, 17, 21, 23, 25, 27, 29

7.5 Solving Initial Value Problems

The method of Laplace transforms leads to the solution of an initial value problem *without* first finding a general solution.

Ex. (ex1, p403)

Ex. (ex2, p404)

Ex. (ex3, p405)

Ex. (ex4, p406)

Ex. (ex5, p408)

Homework

7.5 (p409-410): 1, 3, 5, 7, 11, 13, 25, 27, 35, 37

7.6

7.7

7.8 Impluses and the Dirac Delta Function

Def 7.7. The **Dirac delta function** $\delta(t)$ is characterized by the following two properties:

$$\delta(t) = \begin{cases} 0, & t \neq 0, \\ \text{“infinite,”} & t = 0, \end{cases} \tag{7.8.1}$$

and

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \tag{7.8.2}$$

We have

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \tag{7.8.3}$$

In particular, let $u(t)$ be the unit step function, then

$$\int_{-\infty}^{\infty} \delta(t-a)dt = u(t-a) := \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

If a force $\mathcal{F}(t)$ is applied shortly from time t_0 to time t_1, then the **impulse** due to $\mathcal{F}$ is

$$\text{Impulse} = \int_{t_0}^{t_1} \mathcal{F}(t)dt = \int_{t_0}^{t_1} m\frac{dv}{dt}dt = mv(t_1) - mv(t_0)$$

Since mv represents the momentum, **the impulse equals the changes in momentum.**

If we use a hammer to strike an object by force $\mathcal{F}_n(t)$ to change a unit momentum in a very short period of time $[t_0, t_n]$ where $t_n \to t_0$ as $n \to \infty$, then $\int_{-\infty}^{\infty} \mathcal{F}_n(t)dt = 1$ and $\mathcal{F}_n$ approaches a limiting "function" function $\delta(t)$.

The Laplace transform of $\delta(t-a)$ for $a \geq 0$ is

$$\mathcal{L}\{\delta(t-a)\}(s) = \int_0^{\infty} e^{-st}\delta(t-a)dt = e^{-as}$$

Ex. Example on page 435: $x'' + x = \delta(t)$; $x(0) = 0$, $x'(0) = 0$.

Ex. (ex1, p437)

Homework

7.8 (p438-440): 1, 5, 9, 11, 13, 15, 17, 19