4 Linear Transformations

The operations “+” and “·” provide a linear structure on vector space $V$. We are interested in some mappings (called linear transformations) between vector spaces $L : V \rightarrow W$, which preserves the structures of the vector spaces.

4.1 Definition and Examples

1. Demonstrate: A mapping between two sets $L : V \rightarrow W$.

   **Def.** Let $V$ and $W$ be vector spaces. A mapping $L : V \rightarrow W$ is called a **linear transformation** iff
   
   $$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2), \text{ for all } v_1, v_2 \in V, \alpha, \beta \in \mathbb{R}.$$

   **Equivalent Condition:** $L$ is a linear transformation iff
   
   $$L(v_1 + v_2) = L(v_1) + L(v_2) \quad L(\alpha v) = \alpha L(v).$$

   **Question:** How does a linear map look like?

   **Ex.** Linear transformations on $\mathbb{R}^1$. ($L : \mathbb{R}^1 \rightarrow \mathbb{R}^m$)

   **Ex.** Several examples on $\mathbb{R}^2$: (Show the graphs for a-d, check conditions for a,b)
   
   (a) $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $L(v) = -2v$.
   (b) Ex 2 in textbook (p177 in 7th ed), $L(x) = x_1 e_1$.
   (c) Ex 3 in textbook (p177 in 7th ed), $L(x) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$.
   (d) Ex 4 in textbook (p178 in 7th ed), $L(x) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$.
   (e) $L : \mathbb{R}^2 \rightarrow \mathbb{R}^1$, $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2$.

   **Ex.** Identity map.

   **Ex.** For any $A \in \mathbb{R}^{m \times n}$, define $L_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by
   
   $$L_A(v) := Av \text{ for } v \in \mathbb{R}^n.$$

   Check that $L_A$ is a linear transformation.
Ex. (Counterexample) \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( L(v) = v + e_1 \). Then \( L \) is NOT a linear transformation.

Ex. (Counterexample) \( L : \mathbb{R}^2 \to \mathbb{R}^1 \) defined by \( L(x) = \sqrt{x_1^2 + x_2^2} \). Then \( L \) is NOT a linear transformation.

Ex. Ex 9 (p180 in 7th ed), \( L : C[a,b] \to \mathbb{R}^1 \), defined by \( L(f) := \int_a^b f(x) \, dx \).

Ex. \( L : P_n \to P_{n-1} \) defined by \( L(f)(x) = f'(x) \).

• Linear transformations send subspaces to subspaces.
• HW 12, p183. If \( L : V \to W \) is a linear transformation, then

\[
L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 L(v_1) + \alpha_2 L(v_2) + \cdots + \alpha_n L(v_n).
\]

2. The Image and Kernel.

Def. Let \( L : V \to W \) be a linear transformation. The kernel of \( L \) is

\[
\ker(L) = \{ v \in V \mid L(v) = 0_W \}, \quad \text{(So } \ker(L) \subseteq V \text{)}.
\]

Let \( S \) be a subspace of \( V \). The image of \( S \) is

\[
L(S) = \{ L(v) \mid v \in S \}, \quad \text{(So } L(S) \subseteq W \text{)}.
\]

\( L(V) \): The image of \( V \) is called the range of \( L \).

Ex. The kernel and images of \( L \) are subspaces.

Ex. Let \( A \in \mathbb{R}^{m \times n} \). Let \( L_A : \mathbb{R}^n \to \mathbb{R}^m \) be defined by \( L_A(x) := Ax \). The kernel of \( L_A \) is exactly \( N(A) \). The range of \( L_A \) is

\[
L_A(\mathbb{R}^n) = \{ Ax \mid x \in \mathbb{R}^n \},
\]

which is exactly the column space of \( A \).

Ex. HW 17b, (p184 in 7th ed). Find the kernel and the range of linear operator \( L \) on \( \mathbb{R}^3 \), where \( L(x) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \).

4.1.1 Homework

Sect 4.1.
6th ed 1ade, 3, 6bd, 15, 16ac.
7th ed 1ade, 3, 6bd, 16, 17ac.
4.2 Matrix Representations of Linear Transformations

1. All linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are of the form $L(x) = Ax$ for some $A$.

**Thm 4.1.** Given a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$, there is $A \in \mathbb{R}^{m \times n}$, such that

$$L(x) = Ax, \quad \text{i.e.} \quad L(x) = L_A(x).$$

In fact,

$$A = \begin{bmatrix} L(e_1), L(e_2), \cdots, L(e_n) \end{bmatrix}.$$ 

$A$ is called the standard matrix representation of $L$.

Proof of the theorem.

**Ex.** Ex 1, pp 185 in 7th ed.

**Ex.** Ex 2, p186 in 7th ed.

**Ex.** (skip)

$L : \mathbb{R}^2 \to \mathbb{R}^3, \quad L(x) = \begin{bmatrix} x_1 - x_2 \\ x_1 + 2x_2 \\ -x_1 \end{bmatrix}$.

**Ex.** (skip)

$L : \mathbb{R}^2 \to \mathbb{R}^1, \quad L(x) = x_1 + x_2$

2.

**Thm 4.2.** If $E = [v_1, \cdots, v_n]$ is a basis of $V$, and $F = [w_1, \cdots, w_m]$ is a basis of $W$, for each linear transformation $L : V \to W$, there is $A \in \mathbb{R}^{m \times n}$ such that $[L(v)]_F = A[v]_E$ for $v \in V$. In fact,

$$A = \begin{bmatrix} [L(v_1)]_F, \cdots, [L(v_n)]_F \end{bmatrix}.$$ 

(Refer to Fig 4.2.2 in p188)

**Ex.** Example 4, pp 188 in 7th ed.

**Ex.** Let $F = [b_1, b_2]$ be a basis of $\mathbb{R}^2$, where

$$b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

Find the matrix $A$ representing

$L : \mathbb{R}^3 \to \mathbb{R}^2, \quad L(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = (x_1 + 2x_2)b_1 + (3x_1 + 4x_2)b_2.$

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relative to the standard basis $E$ of $\mathbb{R}^3$ and the basis $F$ of $\mathbb{R}^2$.

3. (Important) The above theorem need to compute coordinate vectors. In practical, we use the following result:

**Thm 4.3.** Let $A$ be the matrix representing $L : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the bases $E = [u_1, \ldots, u_n]$ of $\mathbb{R}^n$ and $F = [b_1, \ldots, b_m]$ of $\mathbb{R}^m$, Then the RREF of $[b_1, \ldots, b_m \mid L(u_1), \ldots, L(u_n)]$ is exactly $[I \mid A]$.

**Ex.** Revisit the previous example.

**Ex.** Ex 6 (p190 in 7th ed).

**Ex.** Application 1 (p191 in 7th ed). Computer graphics and animation.

### 4.2.1 Homework

Sect 4.2 6th ed: 2, 4, 6, 16a, 7th ed: 2, 4, 6, 18a

### 4.3 Similarity

1. A square matrix $B \in \mathbb{R}^{n \times n}$ is similar to $A \in \mathbb{R}^{n \times n}$, iff there is a non-singular matrix $S \in \mathbb{R}^{n \times n}$, such that $B = S^{-1}AS$.

   (a) $A$ is similar to $A$ itself:
   $$A = I_n^{-1}AI_n.$$

   (b) If $A$ is similar to $B$, then $B$ is similar to $A$:
   $$B = S^{-1}AS \implies A = SBS^{-1} = (S^{-1})^{-1}B(S^{-1}).$$

   (c) If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$:

   So similarity is an equivalent relationship.

2. Similarity is important in representing a linear transformation by different bases.

   **Question:** Let $E$ be the standard basis in $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ has the standard matrix representation $L(x) = Ax$. If $F$ is another basis of $\mathbb{R}^n$, what is the matrix representation $B$ of $L$ with respect to $F$ (That is, $[L(v)]_F = B[v]_F$)?

   **Answer:** Let $U$ be the transition matrix from $F$ to $E$.

   $$[L(v)]_E = A[v]_E \implies U[L(v)]_F = AU[v]_F \implies [L(v)]_F = U^{-1}AU[v]_F.$$

   So $B = U^{-1}AU$. 

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**Thm 4.4.** Two square matrices $A$ and $B$ are similar, if and only if both are representing a same linear transformation in different bases.

**Ex.** Example 2 in the textbook (pp204 in 7th ed). **Method 1:** Matrix Representation Theory. **Method 2:** Transition matrix.

- The importance of changing bases: to simplify linear transformations.

**Ex.** problem 4 (pp205 in 7th ed).

**Ex.** problem 9 (pp206 in 7th ed).

### 4.3.1 Homework

Sect 4.3 1ae, 2, 7, 11, 12