3.3 (III-15) Factor Group Computations, Simple Groups, and Solvable Groups

3.3.1 Factor Group Computations

Let N be a normal subgroup of G. The relations between the group G, the subgroup N, and the factor group G/N are shown in Figure 15.4. Think about the example that $G = \mathbf{R}^2$, $N = \{(0, y) \mid y \in \mathbf{R}\}$. Then a coset $a + N = \{(a, y) \mid y \in \mathbf{R}\}$.

There is no universal way to classify the factor group. However, we can solve some easy cases.

Ex 3.39. If a group G has the identity e, then $G/\{e\} \simeq G$.

Thm 3.40. *A factor group of a cyclic group is cyclic.*

Proof. [6] Refer to the proof in the textbook. We give another interpretation:

Let G/N be a factor group of a cyclic group G. Then N is a normal subgroup. So $N = \ker(\phi)$ for a group homomorphism $\phi : G \to G'$. Thus $G/N \simeq \phi[G]$. If G is generated by a, then $\phi[G]$ is generated by $\phi(a)$. So $\phi[G]$ and G/N is cyclic. □

Similarly,

Thm 3.41. *A factor group of an abelian group is abelian.*

Note that a factor group of a *nonabelian* group can also be abelian.

Ex 3.42 (Ex 15.4, p.145). When $n > 2$, the symmetric group S_n is nonabelian, but the factor group $S_n/A_n \simeq \mathbf{Z}_2$ is abelian.

Thm 3.43. *If H_i is a normal subgroup of G_i, then the factor group*

$$\left(\prod_{i=1}^{n} G_i\right) / \left(\prod_{i=1}^{n} H_i\right) \simeq \prod_{i=1}^{n} (G_i/H_i)$$

The theorem includes the case in Theorem 15.8 (p.147). You can try to construct the isomorphism explicitly.

[6] 1st HW: 19, 20, 22, 28, **40**

3.3.2 Simple Groups

Def 3.44. A group G is simple if it is nontrivial and it has no proper nontrivial normal subgroups. That is, $|G| > 1$, and the *only* normal subgroups of G are $\{e\}$ and G itself.

Thm 3.45. *The alternative group A_n is simple for $n \geq 5$.*

The classification of all finite simple groups are done around 1980. It is a milestone in group theory.

Thm 3.46. *Let $\phi : G \to G'$ be a group homomorphism. If N is a normal subgroup of G, then $\phi[N]$ is a normal subgroup of $\phi[G]$. Also, if N' is a normal subgroup of G', then $\phi^{-1}[N']$ is a normal subgroup of G.*

3.3.3 The Center and Commutator Subgroups

Every group G has two important normal subgroups.

Def 3.47. The center of a group G is

$$Z(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

It contains the elements that commutes with *all* elements of G.

Ex 3.48. The center $Z(G)$ is a normal subgroup of G.

Def 3.49. The *commutator subgroup* of G is the group C generated by *all elements* of the set

$$\{aba^{-1}b^{-1} \mid a, b \in G\}.$$

We use $C := [G, G]$ to represent the commutator subgroup.

Thm 3.50. *The commutator subgroup C is a normal subgroup of G. Moreover, if N is a normal subgroup of G. Then G/N is abelian if and only if $C \leq N \leq G$.*

3.3.4 Solvable Groups

Other than simple group, another important type of groups is called the solvable group.

For a group G, we define the following groups

$$\begin{aligned} G^{(0)} &:= G, \\ G^{(1)} &:= [G^{(0)}, G^{(0)}], \qquad \text{the commutator subgroup of } G^{(0)}, \\ G^{(2)} &:= [G^{(1)}, G^{(1)}], \qquad \text{the commutator subgroup of } G^{(1)}, \\ &\vdots \end{aligned}$$

Then we get a sequence of subgroups $G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots$, where $G^{(i+1)}$ is a normal subgroup of $G^{(i)}$, and $G^{(i)}/G^{(i+1)}$ is abelian.

Def 3.51. A group G is a *solvable group*, if $G^{(m)} = \{e\}$ for certain $m \in \mathbf{N}$.

Ex 3.52. Let U be the set of all 2×2 nonsingular real upper triangular matrices. Then

$$\begin{aligned} U^{(0)} = U &= \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbf{R},\ a, c \neq 0 \right\}, \\ U^{(1)} &= \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbf{R} \right\}, \\ U^{(2)} &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \end{aligned}$$

Therefore, U is a solvable group.

Thm 3.53. *The subgroups and factor groups of a solvable group are solvable.*

Solvable groups are useful in solving the roots of polynomials, in finite group theory, in Lie groups, in representation theory, etc.

3.3.5 Homework, Sect 15, p.151-p.154

1st 19, 20, 22, 28, **40**.

2nd 14, 30, 31, **35**, 36.