## 4.3 36. Sylow Theorems and Applications

The structures of finite abelian groups are well classified. The structures of finite *nonabelian* groups are much more complicate (Think about $S_n$, $A_n$, $D_n$, etc). Sylow theorems are very useful in studying finite nonabelian groups. Here we survey the classical results of Sylow theorems and apply them to examples.

**Def 4.30.** Let $p$ be a fixed prime. A group $G$ is a *p-group* if every element in $G$ has order a power of $p$. A subgroup of a group $G$ is a *p-subgroup* of $G$ if the subgroup is itself a $p$-group.

**Thm 4.31 (Cauchy's Theorem).** *Let $G$ be a finite group. Let $p$ be a prime factor of $|G|$. Then $G$ has a subgroup of order $p$.*

**Cor 4.32.** A finite group $G$ is a $p$-group if and only if $|G|$ is a power of $p$.

*Proof by Cauchy's Theorem.* If $|G|$ is a power of $p$, then the order of every element $g$ of $G$ divides $|G|$. So the order of $g$ must be a power of $p$. Thus $G$ is a $p$-group.

If $|G|$ is not a power of $p$, then $|G|$ has another prime factor $q$. By Cauchy's Theorem, there is a subgroup $H$ of $G$ of order $q$. Any nonidentity element in $H$ has order $q$. So $G$ is not a $p$-group. $\square$

**Thm 4.33 (First Sylow Theroem).** *Let $G$ be a finite group with $|G| = p^n m$ where $n \geq 1$ and the prime $p$ does not divide $m$. Then*

1. *$G$ contains a subgroup of order $p^i$ for each $i$ where $1 \leq i \leq n$.*

2. *Every subgroup $H$ of $G$ of order $p^i$ is a normal subgroup of a subgroup of order $p^{i+1}$ for $1 \leq i < n$.*

**Ex 4.34.** $S_4$ has order $4! = 24 = 2^3 \cdot 3$. By the first Sylow theorem, $S_4$ must contain subgroups of orders $2^i$ $(1 \leq i \leq 3)$ and 3. In Example 8.10, $D_4$ is realized as a subgroup of $S_4$ (see page 80). We can easily find subgroups of $S_4$ of order 8 ($D_4$), 4, 2. The subgroup of $S_4$ generated by $(1, 2, 3)$ is a subgroup of order 3.

**Ex 4.35.** $S_6$ has order $6! = 720 = 2^4 \cdot 3^2 \cdot 5$. By the first Sylow theorem, $S_6$ must contain subgroups of orders $2^i$ $(1 \leq i \leq 4)$, $3^j$ $(1 \leq j \leq 2)$, and 5. Every subgroup of $S_6$ of order 4 must be a normal subgroup of certain subgroup of $S_6$ of order 8.

**Ex 4.36.** Every finite $p$-group is solvable.

**Def 4.37.** A *Sylow $p$-subgroup $P$* of a group $G$ is a maximal $p$-subgroup of $G$.

If $G$ is a finite group and $|G| = p^n m$ where $n \geq 1$ and the prime $p$ does not divide $m$, then a Sylow $p$-subgroup of $G$ is exactly a subgroup of $G$ of order $p^n$.

**Ex 4.38.** With the correspondence in Example 8.10 (page 80), $D_4$ is a Sylow 2-subgroup of $S_4$. In $S_6$, a Sylow 2-subgroup has order 16; a Sylow 3-subgroup has order 9; a Sylow 5-subgroup has order 5.

**Thm 4.39 (Second Sylow Theorem).** *Let $p$ be a fixed prime factor of a finite group $G$. Then all Sylow $p$-subgroups of $G$ are conjugate to each other. In other words, if $P_1$ and $P_2$ are both Sylow $p$-subgroups of $G$, then there exists $g \in G$ such that $P_2 = gP_1g^{-1}$.*

**Ex 4.40.** The cyclic subgroups $\langle(1, 2, 3)\rangle$ and $\langle(1, 4, 2)\rangle$ are both Sylow 3-subgroups of $S_4$. They are conjugate to each other.

**Thm 4.41 (Third Sylow Theorem).** *If $G$ is a finite group and $p$ divides $|G|$, then the number of Sylow $p$-subgroups is congruent to 1 modulo $p$ and divides $|G|$.*

**Ex 4.42.** Let $N$ be the number of the Sylow 3-subgroups of $S_4$. Then $N \equiv 1 \pmod 3$ and $N$ divides $|S_4| = 24$. $N$ can be 1 or 4. In fact, there are 4 Sylow 3-subgroups of $S_4$:

$$\langle(1, 2, 3)\rangle, \quad \langle(1, 2, 4)\rangle, \quad \langle(1, 3, 4)\rangle, \quad \langle(2, 3, 4)\rangle.$$

**Ex 4.43 (Ex 36.13, p.326).** If $G$ is a group with $|G| = 15$, then $G$ contains a normal subgroup of order 5. So $G$ is solvable and is not simple.

### 4.3.1 Homework, Section 36, p.326-p.327

2, 3, 6, 11, 13.