2.4 The Action of a Group on a Set

Def. *An* **action** *of a group G on a set S is a function $G \times S \to S$, $(g, x) \mapsto gx$, such that for all $x \in S$ and $g_1, g_2 \in G$:*

$$ex = x \quad \text{and} \quad (g_1 g_2)x = g_1(g_2 x).$$

We say that G **acts on the set** *S.*

Equivalently, an action of G is a group homomorphism $\phi : G \to A(S)$, where $A(S)$ is the group of permutations of S.

Ex. *The symmetric group S_n acts on the set $I_n = \{1, 2, \cdots, n\}$ by $(\sigma, x) \mapsto \sigma(x)$.*

Ex.

1. *Let G be a group and H a subgroup. An action of the group H on the set G is $(h, x) \mapsto hx$ (product in G). This action is called a* **left translation***.*
2. *Let K be another subgroup of G and S the set of all left coesets of K in G. Then H acts on S by translation: $(h, xK) \mapsto hxK$.*

Ex.

1. *Let G be a group and H a subgroup. Another action of the group H on the set G is $(h, x) \mapsto hxh^{-1}$. This action of $h \in H$ on G is called* **conjugation** *by h, and hxh^{-1} is called a* **conjugate** *of x.*
2. *H acts on the set S of all subgroups of G by conjugation: $(h, K) \mapsto hKh^{-1}$.*

Thm 2.22. *Let group G act on a set S.*

1. *For every $x \in S$, the set $\mathcal{O}_x := \{gx \mid g \in G\}$ is an* **orbit** *of G on S, called the G-orbit of x.*
2. *For each $x \in S$, $G_x = \{g \in G \mid gx = x\}$ is a subgroup of G. The subgroup G_x is called the* **isotropy group** *of x or the* **stabilizer** *of x.*

Ex.

1. *Group G acts on itself by conjugation. Then the orbit $\{gxg^{-1} \mid g \in G\}$ is called the* **conjugacy class** *of x.*
2. *If a subgroup H acts on G by conjugation the isotropy group $H_x = \{h \in H \mid hxh^{-1} = x\}$ is called the* **centeralizer of** *x* **in** *H and is denoted $C_H(x)$. In particular, $C_G(x)$ is called the* **centralizer** *of x.*

3. *If H acts by conjugation on the set S of all subgroups of G, then the subgroup of H fixing $K \in S$, namely $\{h \in H \mid hKh^{-1} = K\}$, is called the* **normalizer of** K **in** H *and denoted $N_H(K)$. The group $N_G(K)$ is called the* **normalizer** *of K.*

Thm 2.23. *Let G act on S. Then $|\mathcal{O}_x| = [G : G_x]$.*

Cor 2.24. *Let G be a finite group and K a subgroup of G.*

1. *The number of elements in the conjugacy class of $x \in G$ is $[G : C_G(x)]$, which divides $|G|$.*
2. *If all orbits of G are $\mathcal{O}_{x_1}, \cdots, \mathcal{O}_{x_n}$ for $x_i \in G$, then*
$$|G| = \sum_{i=1}^{n} [G : C_G(x_i)].$$
This is the **class equation** *of the finite group G.*
3. *The number of subgroups of G conjugate to K is $[G : N_G(K)]$, which divides $|G|$.*

Prop 2.25 (Cayley)**.** *If G is a group, then there is a monomorphism $G \to A(G)$. Hence every group is isomorphic to a group of permutations. In particular, every finite group is isomorphic to a subgroup of S_n with $n = |G|$.*

Ex. *Let G act on itself by conjugation.*

1. *For each $g \in G$, conjugation by g induces an automorphism of G, called an* **inner automorphism** *of G;*
2. *The induced homomorphism $G \to Aut(G)$ has the kernel*
$$C(G) = \{g \in G \mid gx = xg \;\; \text{for all} \;\; x \in G\}.$$

Ex. *Let H be a subgroup of a group G and let G act on the set S of all left cosets of H in G by left translation. Then the kernel of the induced homomorphism $G \to A(S)$ is contained in H.*

Prop 2.26. *If $[G : H] = n$ and H contains no nontrivial normal subgroup of G, then G is isomorphic to a subgroup of S_n.*

(proof)

Prop 2.27. *If G is a finite group and $[G : H] = p$, where p is the smallest prime dividing $|G|$, then H is normal in G.*

Proof. Let G act on the set S of left cosets of H in G. Then $|S| = p$. The kernel K of the induced group homomorphism $G \to A(S)$ is in H. Then $A(S) \simeq S_p$, and thus G/K is isomorphic to a subgroup of S_p. So $|G/K|$ divides $\gcd(|G|, |S_p|) = \gcd(|G|, p!) = p$. Therefore, $|G/K| = p$. Then $K = H$ and $H \lhd G$. □