2.6 Groups of Small Order

Thm 2.38. *Let $|G| = pq$, where $p > q$ are primes. Then either $G \simeq \mathbb{Z}_{pq}$ or $G = \langle a, b\rangle$, where*

$$b^p = 1, \qquad a^q = 1, \qquad aba^{-1} = b^m,$$

and $m^q \equiv 1 \mod p$ but $m \not\equiv 1 \mod p$. If $q \mid p - 1$, then the second case cannot occur.

Proof. By Cauchy's theorem or Sylow theorem, G contains a subgroup $H = \langle b\rangle$ of order p and a subgroup $K = \langle a\rangle$ of order q. Then H has index q in G. By Proposition 2.27, $H \lhd G$. The number c of K's conjugates is $1 + kq$ for some $k \geq 0$. As above, either $c = 1$ or $c = p$.

1. If $c = 1$, then $K \lhd G$ and $G \simeq H \times K$, and so $G \simeq \mathbb{Z}_p \times \mathbb{Z}_q \simeq \mathbb{Z}_{pq}$.
2. In case $c = kq + 1 = p$, then $q \mid p - 1$, and $K \ntrianglelefteq G$. Since $H \lhd G$, $aba^{-1} = b^m$ for some m; furthermore, we may assume that $m \not\equiv 1 \mod p$ lest we return to the abelian case. Then $a^j ba^{-j} = b^{m^j}$ by induction on j. In particular, if $j = q$, then $m^q \equiv 1 \mod p$.

□

Cor 2.39. *If p is an odd prime, then every group of order $2p$ is either $\mathbb{Z}_{2p}$ or D_p.*

Prop 2.40. *There are exactly two distinct nonabelian groups of order 8: the quaternion group Q_8 and the dihedral group D_4.*

Proof. A nonabelian group G of order 8 has no element of order 8, and not every nonidentity element has order 2 (HW). Therefore, G has an element a of order 4. Then $[G : \langle a\rangle] = 2$. So $\langle a\rangle \lhd G$. Choose $b \in G\backslash\langle a\rangle$. Then $b^2 \in \langle a\rangle$. The order of b is either 2 or 4, so that $b^2 = e$ or $b^2 = a^2$. Since G is nonabelian, $bab^{-1} \neq a$ so that $bab^{-1} = a^3$. Only two possibilities remain:

Q_8: $a^4 = e$, $b^2 = a^2$, $bab^{-1} = a^3$.

D_4: $a^4 = e$, $b^2 = e$, $bab^{-1} = a^3$.

□

Prop 2.41. *Every nonabelian group G of order 12 is isomorphic to either A_4, D_6, or*

$$T := \langle a, b \mid a^6 = e,\ b^2 = a^3 = (ab)^2\rangle.$$

The proof is skipped.

Let $\mathcal{Q}(n)$ denote the number of groups of order n up to isomorphism. For small n:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathcal{Q}(n)$	1	1	1	2	1	2	1	5	2	2	1	5	1	2	1

n	16	17	18	19	20	21	22	24	25	26	27	28	30	32	64
$\mathcal{Q}(n)$	14	1	5	1	5	2	2	15	2	2	5	4	4	51	267