## 2.7 Nilpotent and Solvable Groups

### 2.7.1 Nilpotent Groups

The center $C(G)$ of a group $G$ is a normal subgroup. Define $C_0(G) = \langle e \rangle$, and $C_i(G)$ the inverse image of $C(G/C_{i-1}(G))$ under the canonical projection $G \to G/C_{i-1}(G)$, for $i = 1, 2, \cdots$. There is the **ascending central series** of $G$:

$$C_0(G) = \langle e \rangle \ \le \ C_1(G) = C(G) \ \le \ C_2(G) \ \le \ \cdots \tag{2.3}$$

In general, if $C_i \le C_{i+1} \le G$, $C_i \lhd G$ and $C_{i+1}/C_i \lhd G/C_i$, then $C_{i+1} \lhd G$. Therefore, $C_i(G) \lhd G$ for all $i$, by induction. (Excise) For $i = 1, 2, \cdots$,

$$C_i(G) = \{x \in G \mid xyx^{-1}y^{-1} \in C_{i-1}(G) \text{ for all } y \in G\}. \tag{2.4}$$

**Def.** *A group $G$ is* **nilpotent** *if $C_n(G) = G$ for some $n$.*

Every abelian group $G$ is nilpotent since $C_1(G) = G$. Every subgroup or homomorphic image of a nilpotent group is nilpotent.

**Thm 2.42.** *Every finite $p$-group $G$ is nilpotent.*

*Proof.* Every nontrivial quotient group of $G$ is a finite $p$-group. Therefore, if $G/C_i(G)$ is nontrivial, then $C(G/C_i(G))$ is nontrivial (Lemma 2.31) so that $C_i(G) \lneqq C_{i+1}(G)$. So $C_n(G) = G$ for some $n$. □

**Thm 2.43.** *If $G = \prod_{j=1}^{k} G_j$, then $G$ is nilpotent if and only if all $G_j$ are nilpotent.*

*Proof.* Use $C_i(G) = \prod_{j=1}^{k} C_i(G_j)$, which can be proved by induction or by (2.4). □

**Lem 2.44.** *Assume $G$ is nilpotent. If $H \lneqq G$, then $H \lneqq N_G(H)$.*

*Proof.* Let $n$ be the largest integer such that $C_n(G) \le H$. Choose $a \in C_{n+1}(G) \backslash H$. Then for $h \in H$, $aha^{-1}h^{-1} \in C_n \le H$. Thus $aha^{-1} \in H$. Thus $a \in N_G(H) \backslash H$. □

**Thm 2.45.** *A finite group $G$ is nilpotent if and only if it is the direct product of its Sylow subgroups.*

*Proof.* If $G$ is a direct product of Sylow subgroups, then $G$ is nilpotent [Theorem 2.42 and Theorem 2.43]. Conversely, suppose $G$ is nilpotent. For every Sylow subgroup $P$ of $G$, $N_G(N_G(P)) = N_G(P)$ [Proposition 2.36]. Therefore, $N_G(P) = G$ [Lemma 2.44]. Then $P \lhd G$. Suppose that $|G|$ has distinct prime factors $p_1, p_2, \cdots, p_k$. For each $p_i$ there is a unique Sylow $p_i$-subgroup $P_i \lhd G$ [Second Sylow Theorem]. Then $\langle P_1, P_2, \cdots, P_k \rangle = P_1 P_2 \cdots P_k$. Moreover, $P_i \cap (P_1 P_2 \cdots P_{i-1} P_{i+1} \cdots P_k) = \{e\}$ by investigating the possible orders of its elements. Therefore, $P_1 P_2 \cdots P_k = P_1 \times P_2 \times \cdots \times P_k$. Thus $G = P_1 \times P_2 \times \cdots \times P_k$ by comparing their orders. □

**Cor 2.46.** *If $G$ is a finite nilpotent group, then $G$ contains a subgroup of order $m$ for any factor $m$ of $|G|$.*

### 2.7.2 Solvable Groups

**Def.** *Let $G$ be a group. For $a, b \in G$, the element $aba^{-1}b^{-1}$ is called a* **commutator**. *The subgroup of $G$ generated by all commutators:*

$$G' := \langle aba^{-1}b^{-1} \mid a, b \in G\rangle$$

*is called the* **commutator subgroup** *of $G$.*

**Thm 2.47.** *$G' \lhd G$. Moreover, $G'$ is the smallest normal subgroup such that $G/G'$ is abelian. Precisely, $N \lhd G$ and $G/N$ is abelian iff $G' \le N \le G$.*

*Proof.* Let $f : G \to G$ be any automorphism. Then $f(G') \le G'$ since

$$f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} \in G'.$$

In particular, for any $a \in G$, if $f(g) = aga^{-1}$ for $g \in G$, then $aG'a^{-1} \le G'$. Hence $G' \lhd G$. Since $abG' = ab(b^{-1}a^{-1}ba)G' = baG'$, $G/G'$ is abelian.

If $N \lhd G$ and $G/N$ is abelian, then $xyN = yxN$ for all $x, y \in G$; then $x^{-1}y^{-1}xy \in N$ for all $x, y \in G$; then $G' \le N$. The converse is easy. □

**Def.** *For a group $G$, let $G^{(0)} := G$ and $G^{(i)} := (G^{(i-1)})'$ for $i = 1, 2, \cdots$. Then*

$$G^{(0)} = G \;\ge\; G^{(1)} = G' \;\ge\; G^{(2)} \;\ge\; \cdots,$$

*where $G^{(i)}$ is called the $i$-th* **derived subgroup** *of $G$.*

(Exercise) It can be shown that $G^{(i)} \lhd G$ for all $i$.

**Def.** *A group $G$ is said to be* **solvable** *if $G^{(n)} = \langle e\rangle$ for some $n$.*

**Prop 2.48.** *Every nilpotent group is solvable.*

*Proof.* Since $C_i(G)/C_{i-1}(G)$ is abelian, we have $C_i(G)' \le C_{i-1}(G)$. If $C_n(G) = G$, then $G^{(n)} = (C_n(G))^{(n)} \le (C_{n-1}(G))^{(n-1)} \le \cdots \le C_0(G) = \langle e\rangle$. □

**Thm 2.49.**

1. *Every subgroup or homomorphic image of a solvable group is solvable.*
2. *If $N \lhd G$, and $N$ and $G/N$ are solvable, then $G$ is solvable.*

*Proof.*

1. Suppose that $G$ is solvable. If $H \leq G$, then $H^{(i)} \leq G^{(i)}$. So $H$ is solvable. If $f : G \to H$ is a homomorphism, then $f(G^{(i)}) = f(G)^{(i)}$. So $f(G)$ is solvable.

2. Assume that $N \lhd G$, and $N$ and $G/N$ are solvable. Then

$$\begin{aligned} N &= N^{(0)} > N^{(1)} > \cdots > N^{(s)} = \langle e \rangle \qquad \text{and} \\ G/N &= (G/N)^{(0)} > (G/N)^{(1)} > \cdots > (G/N)^{(t)} = \{N\} \end{aligned}$$

   for some $s, t \in \mathbb{N}$. Then $G^{(i)}N/N = (G/N)^{(i)}$ so that $G^{(t)} \leq N$. So $G^{(t+s)} \leq N^{(s)} = \langle e \rangle$.

□

**Cor 2.50.** *The symmetric group $S_n$ for $n \geq 5$ is not solvable.*

A generalization of the Sylow theorems for finite solvable groups is below:

**Prop 2.51** (P. Hall)**.** *Let $G$ be a finite solvable group of order $mn$, with* $\gcd(m, n) = 1$*. Then*

1. *$G$ contains a subgroup of order $m$; conversely, if $G$ is a finite group such that whenever $|G| = mn$ with* $\gcd(m, n) = 1$*, $G$ has a subgroup of order $m$, then $G$ is solvable.*

2. *any two subgroups of $G$ of order $m$ are conjugate;*

3. *any subgroup of $G$ of order $k$, where $k \mid m$, is contained in a subgroup of order $m$.*

The proof is skipped. P. Hall also shown that: If $G$ is a finite group having a $p$-complement for every prime factor $p$ of $|G|$, then $G$ is solvable.

Every finite group of odd order is solvable [conjectured by Burnside, proved by W. Feit and J. Thompson in 1963].