2.8 Normal and Subnormal Series

Def. *A* **subnormal series** *of a group G is a chain of subgroup*

$$G = G_0 > G_1 > \cdots > G_n$$

such that $G_i \triangleright G_{i+1}$ for all i. If in addition $G \triangleright G_i$ for all i, then it is called a **normal series***. The* **factors** *of the series are the quotient groups G_i/G_{i+1}. The* **length** *of the series is the number of strict inclusions.*

Ex. *The derive series $G > G^{(1)} > \cdots > G^{(n)}$ is a normal series for any group G. If G is nilpotent, the ascending central series $\langle e\rangle < C_1(G) < \cdots < C_n(G) = G$ is a normal series of G.*

Def. *A subnormal series $G = H_0 > H_1 > \cdots > H_m$ is a* **refinement** *of a subnormal series $G = G_0 > G_1 > \cdots > G_n$ if $G_0, G_1, \cdots, G_n$ is a subsequence of $H_0, H_1, \cdots, H_m$. A refinement is* **proper** *if its length is larger than that of the original series.*

Def. *A subnormal series $G = G_0 > G_1 > \cdots > G_n = \langle e\rangle$ is a* **composition series** *if each factor G_i/G_{i+1} is simple. It is a* **solvable series** *if each factor is abelian.*

Thm 2.52.

1. *Every finite group G has a composition series.*
2. *Every refinement of a solvable series is a solvable series.*
3. *A subnormal series is a composition series iff it has no proper refinements.*

Thm 2.53. *A group G is solvable iff it has a solvable series.*

Thm 2.54. *A group G is solvable iff it has a composition series whose factors are cyclic of prime order.*

Def. *Two subnormal series S and T of a group G are* **equivalent** *if there is a one-to-one correspondence between the nontrivial factors of S and T such that corresponding factors are isomorphic.*

Lem 2.55 (Zassenhaus). *Let A^*, A, B^*, B be subgroups of a group G such that $A^* \triangleleft A$ and $B^* \triangleleft B$. Then*

1. $A^*(A \cap B^*) \triangleleft A^*(A \cap B)$;
2. $B^*(A^* \cap B) \triangleleft B^*(A \cap B)$;
3. $A^*(A \cap B)/A^*(A \cap B^*) \simeq B^*(A \cap B)/B^*(A^* \cap B)$.

Proof. $B^* \lhd B$ implies that $A \cap B^* \lhd A \cap B$. Similarly, $A^* \cap B \lhd A \cap B$. Let $D := (A^* \cap B)(A \cap B^*)$. Then $D \lhd A \cap B$. We will define an epimorphism $f : A^*(A \cap B) \to (A \cap B)/D$ with kernel $A^*(A \cap B^*)$. This proves that$A^*(A \cap B^*) \lhd A^*(A \cap B)$ and $A^*(A \cap B)/A^*(A \cap B^*) \simeq (A \cap B)/D$.

Define $f : A^*(A \cap B) \to (A \cap B)/D$ as follows: If $a \in A^*$, $c \in A \cap B$, let $f(ac) = Dc$. Verify that f is well-defined, f is surjective, and f is an homomorphism. Finally, $ac \in \operatorname{Ker} f$ iff $c \in D$, iff $c = c_1 c_2$ where $c_1 \in A^* \cap B$ and $c_2 \in A \cap B^*$, iff $ac = (ac_1)c_2 \in A^*(A \cap B^*)$. Therefore, $\operatorname{Ker} f = A^*(A \cap B^*)$.

Similarly, $B^*(A^* \cap B) \lhd B^*(A \cap B)$ and $B^*(A \cap B)/B^*(A^* \cap B) \simeq (A \cap B)/D$. The proof is completed. □

Thm 2.56 (Schreier)**.** *Any two subnormal [resp. normal] series of a group G have subnormal [resp. normal] refinements that are equivalent.*

Proof. Let $S_1 : G = G_0 > G_1 > \cdots > G_n$ and $S_2 : G = H_0 > H_1 > \cdots > H_m$ be subnormal [resp. normal] series. Let $G_{n+1} = \langle e \rangle = H_{n+1}$. Denote $G(i,j) := G_{i+1}(G_i \cap H_i)$. Then $G(i,0) = G_i$. We have the following subnormal [resp. normal] refinement of the series S_1:

$$G = G(0,0) > G(0,1) > \cdots > G(0,m) > G(1,0) > G(1,1) > \cdots > G(1,m)$$
$$> G(2,0) > \cdots > G(n-1,m) > G(n,0) > \cdots > G(n,m).$$

Likewise, denote $H(i,j) := H_{i+1}(G_i \cap H_j)$. Then $H(0,j) = H_j$. We have the following subnormal [resp. normal] refinement of the series S_2:

$$G = H(0,0) > H(1,0) > \cdots > H(n,0) > H(0,1) > H(1,1) > \cdots > H(n,1)$$
$$> H(0,2) > \cdots > H(n,m-1) > H(0,m) > \cdots > H(n,m).$$

Applying Lemma 2.55 to G_{i+1}, G_i, H_{j+1}, and H_j,

$$\frac{G(i,j)}{G(i,j+1)} = \frac{G_{i+1}(G_i \cap H_j)}{G_{i+1}(G_i \cap H_{j+1})} \simeq \frac{H_{j+1}(G_i \cap H_j)}{H_{i+1}(G_{i+1} \cap H_j)} = \frac{H(i,j)}{H(i+1,j)}.$$

These provide the one-to-one correspondence of the factors and show that the refinements are equivalent. □

Thm 2.57 (Jordan-Hölder)**.** *If a group G has a composition series, then any two composition series of G are equivalent. So every group having a composition series determines a unique list of simple groups as factors.*

Ex. $\mathbb{Z}_{24}$ *has the following isomorphic composition series:*

$$\{0\} < \langle 8 \rangle < \langle 4 \rangle < \langle 2 \rangle < \mathbb{Z}_{24},$$
$$\{0\} < \langle 12 \rangle < \langle 4 \rangle < \langle 2 \rangle < \mathbb{Z}_{24},$$
$$\{0\} < \langle 12 \rangle < \langle 6 \rangle < \langle 2 \rangle < \mathbb{Z}_{24},$$
$$\{0\} < \langle 12 \rangle < \langle 6 \rangle < \langle 3 \rangle < \mathbb{Z}_{24}.$$