Algebra I Homework Two

Name:

Instruction: In the following questions, you should work out the solutions in a clear and concise manner. You may share the ideas with peers, but you should write down the solutions independently. Three questions will be randomly selected and checked for correctness; they count 50% grades of this homework set. The other questions will be checked for completeness; they count the rest 50% grades of the homework set. Staple this sheet of paper as the cover page of your homework set. The sample answers will be posted on the course website.

1. (Section 1.6) Prove that S_n is generated by the following elements:

   (a) S_n is generated by the $n-1$ transpositions (12), (13), $\cdots$, $(1n)$. [Hint: $(1i)(1j)(1i) = (ij)$]

   S_n is generated by transpositions (ij), and $(ij) = (1i)(1j)(1i)$. So S_n is generated by $(12), (13), \cdots, (1n)$.

   (b) S_n is generated by the $n-1$ transpositions (12), (23), $\cdots$, $(n-1\,n)$. [Hint: $(1j) = (1\,j-1)(j-1\,j)(1\,j-1)$]

   Each of the transpositions (12), (13), $\cdots$, $(1n)$ can be generated by (12), (23), $\cdots$, $(n-1\,n)$ using $(1\,j) = (1\,j-1)(j-1\,j)(1\,j-1)$ and induction. So S_n can be generated by the $n-1$ transpositions (12), (23), $\cdots$, $(n-1\,n)$ by preceding claim.

   (c) S_n is generated by (12) and $(123\cdots n)$. [Hint: $(i\,i+1) = (12\cdots n)(i-1\,i)(12\cdots n)^{-1}$]

   By $(i\,i+1) = (12\cdots n)(i-1\,i)(12\cdots n)^{-1}$ and induction, (12) and $(123\cdots n)$ generate $(i-1\,i)$ for $i = 2, \cdots, n$. Then by the preceding claim, (12) and $(123\cdots n)$ generate the whole group S_n.

2. (Section 1.6) A_n is the only subgroup of S_n of index 2. [Hint: Show that a subgroup of index 2 must contain all 3-cycles of S_n.]

   The case $n = 2$ is obvious. Now assume that $n \geq 3$. Let $H < S_n$ and $[S_n : H] = 2$. If $(1\ 2\ 3) \notin H$, then $(1\ 3\ 2) = (1\ 2\ 3)^2 \in H$ since $|G/H| = 2$. All 3-cycles in S_n are conjugate to $(1\ 3\ 2)$. As a normal subgroup of S_n, H contains all 3-cycles in S_n. Hence $H \geq A_n$. So $H = A_n$.

3. (Section 1.6) Find all normal subgroups of D_n.

   D_n is generated by two elements a and b with $a^n = b^2 = e$ and $ba = a^{-1}b$. Then $ba^k = a^{-k}b$ by induction. All elements of D_n are of the form a^i or a^ib for $i = 0, 1, \cdots, n-1$. Let H be a normal subgroup of D_n.

   If all elements of H are of the form a^i, then $H = \langle a^k \rangle$ for some k dividing n. Such an H is normal in D_n.

   If H contains an element a^kb, then $a^{k+2}b = a(a^kb)a^{-1} \in H$ by normality of H, and $a^{k+2}ba^kb = a^2 \in H$. So $a^{k-2\ell}b = (a^2)^{-\ell}(a^kb) \in H$. Therefore at least one of b and ab is in H. Therefore H is one of the following: $\langle a^2, b \rangle$ (when n is even), $\langle a^2, ab \rangle$ (when n is even), D_n.

   Overall, all normal subgroups of D_n are:

   (a) D_n and $\{e\}$;

   (b) $\langle a^k \rangle$ for k a factor of n;

   (c) $\langle a^2, b \rangle$ and $\langle a^2, ab \rangle$, when n is even.

4. (Section 1.7) In the category $\mathcal{G}$ of groups, show that the group $G_1 \times G_2$ together with the canonical projections $\pi_1 : G_1 \times G_2 \to G_1$ and $\pi_2 : G_1 \times G_2 \to G_2$ is a product for $\{G_1, G_2\}$.

   Given a group H and a family of homomorphisms of groups $\{\varphi_i : H \to G_i \mid i = 1, 2\}$, define $\varphi : H \to G_1 \times G_2$ by $\varphi(h) := (\varphi_1(h), \varphi_2(h))$. Then $\pi_i \circ \varphi(h) = \pi_i((\varphi_1(h), \varphi_2(h))) = \varphi_i(h)$ for $i = 1, 2$. Thus $G_1 \times G_2$ together with π_i $(i = 1, 2)$ is a product for $\{G_1, G_2\}$.

5. (Section 1.7) Every family $\{A_i \mid i \in I\}$ in the category of sets has a coproduct. [Hint: consider $\dot{\cup} A_i = \{(a, i) \in (\cup A_i) \times I \mid a \in A_i\}$ with $A_i \to \dot{\cup} A_i$ given by $a \mapsto (a, i)$. $\dot{\cup} A_i$ is called the *disjoint union* of the sets A_i.]

   Consider $\dot{\cup} A_i = \{(a, i) \in (\cup A_i) \times I \mid a \in A_i\}$ with $\iota_i : A_i \to \dot{\cup} A_i$ given by $a \mapsto (a, i)$. Given a set B and a family of functions $\{\varphi_i : A_i \to B \mid i \in I\}$, define the function $\varphi : \dot{\cup} A_i \to B$ by $(a, j) \mapsto \varphi_j(a)$. Then for $a \in A_i$, $\varphi \circ \iota_i(a) = \varphi((a, i)) = \varphi_i(a)$. So $\dot{\cup} A_i$ is the coproduct of $\{A_i \mid i \in I\}$ in the category of sets.

6. (Section 1.8)

   (a) S_3 is not the direct product of any family of its proper subgroups.

   Suppose on the contrary, $S_3 = \prod_{i \in I} G_i$ for some nontrivial subgroups G_i. By $|S_3| = 6 = 2 \times 3$, we have $|I| = 2$ and $S_3 = G_1 \times G_2$, where $\{|G_1|, |G_2|\} = \{2, 3\}$. Then both G_1 and G_2 are abelian, and so is $S_3 = G_1 \times G_2$. This contradicts the fact that S_3 is non-abelian.

(b) $\mathbf{Z}_{p^n}$ (p prime, $n \geq 1$) is not the direct product of any family of its proper subgroups.

If $\mathbf{Z}_{p^n} = H_1 \times H_2$, then $|H_1| = p^r$ and $|H_2| = p^{n-r}$ for some $1 \leq r \leq n-1$. Let $d := \max\{r, n-r\} \leq n-1$. On one hand, $\mathbf{Z}_{p^n} = \langle x \rangle$ for some element x with $|x| = p^n$. On the other hand, every $y \in H_1 \times H_2$ satisfies that $|y| = p^r$ for some $r \leq d < n$. This is a contradiction.

(c) $\mathbf{Z}$ is not the direct product of any family of its proper subgroups.

Suppose on the contrary, $\mathbf{Z} = H_1 \times H_2$ where H_1 and H_2 are proper subgroups. Then $H_1 = \langle a \rangle$ and $H_2 = \langle b \rangle$ for $a, b \in \mathbf{Z}$. Then $H_1 \cap H_2 = \langle c \rangle$ for $c = lcm(a,b)$. This contradicts the fact that direct product components have trivial intersection. So the statement is true.

7. (Section 1.8) Give an example to show that the weak direct product is not a coproduct in the category of all groups. [Hint: it suffices to consider the case of two factors $G \times H$.]

Consider the weak direct product $\mathbf{Z}_2 \times \mathbf{Z}_3$ of $\mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$ and $\mathbf{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ with canonical inclusions $\iota_1 : \mathbf{Z}_2 \to \mathbf{Z}_2 \times \mathbf{Z}_3$ and $\iota_2 : \mathbf{Z}_3 \to \mathbf{Z}_2 \times \mathbf{Z}_3$. Define $\varphi_1 : \mathbf{Z}_2 \to S_3$ by $\bar{1} \mapsto (12)$, and define $\varphi_2 : \mathbf{Z}_3 \to S_3$ by $\bar{1} \mapsto (123)$. Then both φ_1 and φ_2 are homomorphisms of groups. If there exists a homomorphism of groups $\varphi : \mathbf{Z}_2 \times \mathbf{Z}_3 \to S_3$ such that $\varphi \circ \iota_i = \varphi_i$ for $i = 1, 2$, then $(12), (123) \in Im(\varphi)$, and thus $Im(\varphi) = S_3$, and thus φ is an isomorphism between $\mathbf{Z}_2 \times \mathbf{Z}_3$ and S_3. This is a contradiction because only one of $\mathbf{Z}_2 \times \mathbf{Z}_3$ and S_3 is abelian.

8. (Section 1.9) The group defined by generators a, b and relations $a^n = e$ ($3 \leq n \in \mathbf{Z}^+$), $b^2 = e$ and $abab = e$ is the dihedral group D_n. [See Theorem 1.6.13.]

Let G be the group defined by generators a, b and relations $a^n = e$, $b^2 = e$ and $abab = e$. Let E be the identity element of D_n. Theorem 1.6.13 says that D_n is generated by two elements A and B with $A^n = B^2 = ABAB = E$. By Theorem 1.9.5, there is an epimorphism of groups $f : G \to D_n$. In particular $|G| \geq |D_n| = 2n$. Now in G, $ba = a^{-1}b$. So every element of G is of the form $a^i b^j$ for $0 \leq i \leq n-1$ and $0 \leq j \leq 1$. Hence $|G| \leq 2n = |D_n|$, and thus $|G| = |D_n|$. Then $f : G \to D_n$ is an isomorphism. This proves the claim.

9. (Section 1.9) The operation of free product is commutative and associative: for any groups A, B, C, $A \star B \simeq B \star A$, and $A \star (B \star C) \simeq (A \star B) \star C$.

The definition of free product shows that

$$\begin{aligned} A \star B = B \star A &= \{a_1 \cdots a_k \mid k \in \mathbf{N},\ \ a_i \in A \cup B,\ \ a_i \text{ and } a_{i+1} \text{ are not in the same group}\}, \\ A \star (B \star C) \simeq (A \star B) \star C &\simeq \{a_1 \cdots a_k \mid k \in \mathbf{N},\ \ a_i \in A \cup B \cup C,\ \ a_i \text{ and } a_{i+1} \text{ are not in the same group}\}. \end{aligned}$$

Alternatively, we may show that both $A \star (B \star C)$ and $(A \star B) \star C$ are coproducts of $\{A, B, C\}$ in the family of groups.

10. (Section 1.9) A free group is a free product of infinite cyclic groups.

Let X be a nonempty set, and $F(X)$ the free group generated by X. We prove that $F(X) = \prod_{x \in X}^{*} \langle x \rangle$, where $\langle x \rangle$ denotes the free group generated by x, an infinite cyclic group.

Let $\iota : X \to F(X)$ be the canonical inclusion map. For every $x \in X$, ι induces a homomorphism of groups $\iota_x : \langle x \rangle \to F(X)$ by $x^n \mapsto \iota(x)^n$. Given a group G and a family of homomorphisms of groups $f_x : \langle x \rangle \to G$ for $x \in X$, define $f : X \to G$ by $x \mapsto f_x(x)$. By the definition of free group on X, there is a unique homomorphism of groups $\widetilde{f} : F(X) \to G$ such that $\widetilde{f} \circ \iota(x) = f(x)$ for $x \in X$. Then

$$\widetilde{f} \circ \iota_x(x^n) = \widetilde{f} \circ (\iota(x)^n) = [\widetilde{f} \circ \iota(x)]^n = f(x)^n \qquad \text{for } \ x^n \in \langle x \rangle.$$

This shows that $F(X)$ is the coproduct of $\{\langle x \rangle \mid x \in X\}$ in the category of groups, that is, the free product of $\{\langle x \rangle \mid x \in X\}$.