Chapter 2
Semisimple Lie Algebras

2.1 Killing form

2.1.1 Criterion for Semisimplicity

The Killing form of a Lie algebra $L$ is a bilinear form $\kappa : L \times L \to F$ defined by:

$$\kappa(x, y) = \text{Tr} (\text{ad}x \text{ad}y) \quad \text{for } x, y \in L.$$  

The Killing form is

1. symmetric: $\kappa(x, y) = \kappa(y, x)$, and
2. associative: in the sense that $\kappa([x, y], z) = \kappa(x, [y, z])$ (since $\text{Tr} ([\text{ad}x, \text{ad}y] \text{ad}z) = \text{Tr} (\text{ad}x \text{ad}y \text{ad}z)$.)

The radical of Killing form (or any symmetric bilinear form of $L$) is $S = \{x \in L \mid \kappa(x, y) = 0 \text{ for all } y \in L\}$. Given a basis $\{x_1, \cdots, x_n\}$ of $L$, the dimension of radical is

$$\dim S = n - \text{rank} \left[ \kappa(x_i, x_j) \right]_{n \times n}.$$  

We call $\kappa$ nondegenerate if $\dim S = 0$, i.e., the matrix $[\kappa(x_i, x_j)]_{n \times n}$ is nondegenerate.

3. the radical $S$ of $\kappa$ is an ideal: by associativity of $\kappa$, if $x \in S$ and $y, z \in L$, then

$$\kappa([x, y], z) = \kappa(x, [y, z]) = 0 \implies [x, y] \in S.$$  

Ex. Compute the matrix form of the Killing form $\kappa$ of $\mathfrak{sl}(2, F)$ w.r.t. the basis $\{h, e, f\}$:

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$  

Lem 2.1. Suppose a Lie algebra $L$ has the Killing form $\kappa$, and $I$ is an ideal of $L$. Then:

1. the Killing form of $I$ is $\kappa_I = \kappa|_{I \times I}$;

2. the orthogonal subspace $I^\perp$ of $I$ w.r.t. $\kappa$ is also an ideal of $L$:

$$I^\perp := \{x \in L \mid \kappa(x, y) = 0 \text{ for any } y \in I\}.$$  

(Note that in general $I \cap I^\perp \neq 0$.)
Proof. For \( x, y \in I \), \( \text{ad} \ x \) (resp. \( \text{ad} \ y \)) maps \( L \) to \( I \). Therefore,

\[
\kappa(x, y) = \text{Tr}(\text{ad} \ x \text{ad} \ y) = \text{Tr}(\text{ad} x|_I \text{ad} y|_I) = \kappa|_I(x, y).
\]

For any \( x \in I^\perp \), \( y \in L \), and \( z \in I \),

\[
\kappa([x, y], z) = \kappa(x, [y, z]) = 0.
\]

Therefore, \( I^\perp \) is also an ideal of \( L \).

Corollary 1.29 implies that: (exercise)

A Lie algebra \( L \) is solvable iff the radical of its Killing form contains \([L, L]\).

Now we develop a criterion for \( L \) to be semisimple, i.e., the maximal solvable ideal \( \text{Rad} \ L = 0 \).

Thm 2.2. A Lie algebra \( L \) is semisimple iff its Killing form is nondegenerate.

Proof. If \( L \) is semisimple, then \( \text{Rad} \ L = 0 \). Let \( S \) be the radical of \( \kappa \). Then \( \text{Tr}(\text{ad} \ x \text{ad} y) = 0 \) for any \( x \in S \) and \( y \in L \) (esp. for \( y \in [S, S] \)). By Corollary 1.29 \( S \) is solvable. Therefore, \( S = 0 \).

Conversely, suppose on the contrary, \( S = 0 \) but \( \text{Rad} \ L \neq 0 \). Then the last nonzero term \( I \) in the derived series of \( \text{Rad} \ L \) is a nonzero abelian ideal of \( L \) (exercise). For any \( x \in I \) and \( y \in L \), \( \text{ad} \ x \text{ad} y \) sends \( L \to L \to I \). So the image of \((\text{ad} \ x \text{ad} y)^2\) is in \([I, I]\). Therefore, \((\text{ad} \ x \text{ad} y)^2 = 0\), which implies that \( \text{ad} \ x \text{ad} y \) is nilpotent and \( \kappa(x, y) = \text{Tr}(\text{ad} \ x \text{ad} y) = 0 \). This shows that \( I \subseteq S = 0 \), a contradiction. Hence \( \text{Rad} \ L = 0 \).

Remark. The proof also shows that \( S \subseteq \text{Rad} \ L \). However, the converse need not hold.

Next we explore some applications of the Killing form.

2.1.2 Simple Ideals of Semisimple Lie Algebra

A Lie algebra \( L \) is a direct sum of ideals \( L_1, \ldots, L_t \) if \( L = L_1 \oplus \cdots \oplus L_t \) as vector spaces. Obviously, \([L_i, L_j] = 0\) for \( i \neq j \).

Thm 2.3. Let \( L \) be semisimple with Killing form \( \kappa \). Then

1. \( L \) is a direct sum of some simple ideals: \( L = L_1 \oplus \cdots \oplus L_t \).

2. The Killing form of \( L_i \) is exactly \( \kappa_i = \kappa|_{L_i \times L_i} \). There is an orthogonal direct sum \( \kappa = \kappa_1 \oplus \cdots \oplus \kappa_t \).

3. Every simple ideal of \( L \) coincides with one of the \( L_i \).

4. Every ideal \( I \) of \( L \) is a direct sum of some \( L_i \)'s, which is semisimple. There is a direct sum of ideals \( L = I \oplus I^\perp \) w.r.t. the Killing form.

5. Every homomorphic image of \( L \) is semisimple and isomorphic to a direct sum of some \( L_i \)'s.

6. \( L = [L, L] \).
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Proof. Let $I$ be any ideal of $L$. Then $I^\perp$ and $I \cap I^\perp$ are also ideals of $L$. By Cartan’s Criterion, $I \cap I^\perp$ is solvable. Hence $I \cap I^\perp = 0$ and $L = I \oplus I^\perp$ by dimension counting. Moreover, any ideal $J$ of $I$ is also an ideal of $L$, and hence an orthogonal direct sum component of $L$ w.r.t. $\kappa$. Therefore, $L$ can be decomposed into an orthogonal direct sum of indecomposable nonabelian ideals, aka. simple ideals:

$$L = L_1 \oplus \cdots \oplus L_t, \quad L_i \perp L_j \text{ w.r.t. } \kappa \text{ for } i \neq j.$$ 

Claims 1 and 2 are proved.

If $I$ is any ideal of $L$, then $I = [I, L] = [I, L_1] \oplus \cdots \oplus [I, L_t]$. Each $[I, L_i] \subseteq I \cap L_i$ is either 0 or $L_i$. It immediately implies Claims 3, 4, and 5.

Finally,

$$[L, L] = \bigoplus_i \bigoplus_j [L_i, L_j] = \bigoplus_i [L_i, L_i] = \bigoplus_i L_i = L.$$

Remark. The study of semisimple Lie algebras can be done by exploring the simple Lie algebras.

2.1.3 Derivations

We have shown that ad $L$ is an ideal of Der $L$. When $L$ is semisimple, it turns out that every derivation of $L$ is inner.

Thm 2.4. If $L$ is semisimple, then $\text{ad} \, L = \text{Der} \, L$.

Proof. $A := \text{ad} \, L$ is an ideal of $D := \text{Der} \, L$. So the Killing form $\kappa_A$ is the restriction of $\kappa_D$ to $A \times A$. Since $L$ is semisimple, $Z(L) = 0$ and $A \simeq L/Z(L) \simeq L$. Therefore, $\kappa_A$ is nondegenerate. There is a direct sum of ideals $D = A \oplus A^\perp$ (w.r.t. the Killing form $\kappa_D$). For any $\delta \in A^\perp$ and $x \in L$,

$$0 = [\delta, \text{ad} \, x] = \text{ad} \, (\delta x) \implies \delta x = 0 \text{ for any } x \in L.$$ 

Therefore, $\delta = 0$, $A^\perp = 0$, and $D = A$. \qed

Remark. When $L$ is semisimple, the Lie algebra of Aut $L$ is Der $L = \text{ad} \, L$. If $G$ is a (real or complex) connected Lie group whose Lie algebra $L$ is semisimple, then the Lie algebra of Aut$(G)$ is exactly Der $L = \text{ad} \, L$.

2.1.4 Abstract Jordan Decomposition

Lemma 1.27 shows that Der $L$ contains the semisimple part and the nilpotent part of all its elements. When $L$ is semisimple, Der $L = \text{ad} \, L$. We can write every ad $x \in \text{ad} \, L$ uniquely as

$$\text{ad} \, x = \text{ad} \, x_s + \text{ad} \, x_n,$$

where $x_s, x_n \in L$, ad $x_s$ is semisimple, ad $x_n$ is nilpotent, and ad $x_s$ and ad $x_n$ commute. Then $x = x_s + x_n$ and $[x_s, x_n] = 0$. This is called the abstract Jordan decomposition of $x$ in $L$, and $x_s$ (resp. $x_n$) is called the semisimple part (resp. nilpotent part) of $x$.

The abstract Jordan decomposition is preserved by direct sums (exercise), Lie algebra homomorphisms, and representations (to be proved in the next section).