2.3 Representation of $\mathfrak{sl}(2, F)$

The representations of $\mathfrak{sl}(2, F)$ play an important role in the study of semisimple Lie algebras. In this section, we consider the finite dimensional representations of $L := \mathfrak{sl}(2, F)$, whose standard basis consists of

$$h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

such that

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

Let $V$ be an arbitrary $L$-module. Since $h$ is semisimple in $L$, the preservation of Jordan decomposition implies that $h$ acts diagonally on $V$. So $V$ is a direct sum of eigenspaces $V_\lambda := \{ v \in V \mid h.v = \lambda v \}$.

We let $V_\lambda = 0$ if $\lambda$ is not an eigenvalue of the $h$-action on $V$. Whenever $V_\lambda \neq 0$, we call $\lambda$ a weight of $h$ in $V$ and $V_\lambda$ a weight space.

**Lem 2.13.** If $v \in V_\lambda$, then $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$.

**Proof.** $h.(x.v) = [h, x].v + x.(h.v) = 2x.v + \lambda x.v = (\lambda + 2)x.v$. Similarly for $y$. \qed

Since $\dim V < \infty$ and $V = \bigoplus_{\lambda \in F} V_\lambda$, there exists $\lambda \in F$ such that $V_\lambda \neq 0$ but $V_{\lambda+2} = 0$; this weight $\lambda$ is called a highest weight, and any nonzero vector $v \in V_\lambda$ is called a maximal vector of weight $\lambda$, where $v \neq 0$ but $x.v = 0$. We will see that a highest weight is a nonnegative integer.

Every $L$-module is a direct sum of irreducible submodules. The following theorem completely classifies all irreducible $L$-modules.

**Thm 2.14.** Let $\phi : L \to \mathfrak{sl}(V)$ be an irreducible representation of $L = \mathfrak{sl}(2, F)$, and $\dim V = m + 1 < \infty$. Then there exists a basis $\mathcal{B} = \{v_0, v_1, \ldots, v_m\}$ of $V$, such that $\phi(h), \phi(x)$ and $\phi(y)$ have the following matrix forms relative to the basis $\mathcal{B}$:

$$\phi(h) \overset{\mathcal{B}}{\approx} \begin{bmatrix} m & m-2 & \cdots & -(m-2) \\ & & \ddots & \\ & & & -m \end{bmatrix}, \quad \phi(x) \overset{\mathcal{B}}{\approx} \begin{bmatrix} 0 & m & \cdots \\ & 0 & \ddots \\ & & \ddots \\ & & & 2 \\ & & & 0 \end{bmatrix}, \quad \phi(y) \overset{\mathcal{B}}{\approx} \begin{bmatrix} 0 & 1 & \cdots \\ & 2 & \ddots \\ & & \ddots \\ & & & 0 \end{bmatrix}.$$

In particular,

1. For $m \in \mathbb{N}$, (up to isomorphism) there exists exactly one irreducible $L$-module of dimension $m + 1$, denoted by $V(m)$.

2. $V(m)$ is a direct sum of weight spaces relative to $h$: $V = \bigoplus_{i=0}^{m} V_{m-2i}$ (the irreducibility of $V(m)$ is done in homework). $V(m)$ has the highest weight $m$, and $V(m)$ has a unique (up to nonzero scalar multiplies) maximal vector in $V_m$. 
Proof. Let $\lambda$ be a highest weight and choose a maximal vector $v_0 \in V_{\lambda} - \{0\}$. Set $v_{-1} = 0$ and $v_i = (1/i!)y^i.v_0$ for $i \geq 0$. Lemma 2.13 implies that
\[ h.v_i = (\lambda - 2i)v_i. \quad (2.1) \]
The definition of $v_i$ implies that
\[ y.v_i = (i + 1)v_{i+1}. \quad (2.2) \]
We use induction on $i$ to prove that
\[ x.v_i = (\lambda - i + 1)v_{i-1} \quad \text{for} \quad i \geq 0. \quad (2.3) \]
The case $i = 0$ is obviously true. For $i > 0$,
\[
ix_i = x.y.v_{i-1} = [x, y].v_{i-1} + y.x.v_{i-1} \overset{\text{I.H.}}{=} h.v_{i-1} + (\lambda - i + 2)y.v_{i-2} = (\lambda - 2i + 2)v_{i-1} + (\lambda - i + 2)(i - 1)v_{i-1} = i(\lambda - i + 1)v_{i-1}.
\]
Divide both sides by $i$ to complete the induction process.

The equality $h.v_i = (\lambda - 2i)v_i$ shows that $v_i \in V_{\lambda - 2i}$. Hence the nonzero $v_i$ are linearly independent. By $\dim V < \infty$, there exists $n \in \mathbb{N}$ such that $v_n \neq 0$ but $v_{n+1} = 0$. Then $v_0, v_1, \ldots, v_n$ are linearly independent, and they span a nonzero $L$-submodule of $V$, which must be $V$ itself due to irreducibility of $V$. So $n = m$. By (2.3), $0 = x.v_{m+1} = (\lambda - m)v_m$, so that $\lambda = m$. Overall (2.1), (2.2), and (2.3) lead to the desired matrix forms of $\phi(h)$, $\phi(x)$, and $\phi(y)$ w.r.t. the basis $\{v_0, v_1, \ldots, v_m\}$. \hfill\(\square\)

The structure of any finite dimensional $L$-module can be determined by its weight spaces as follow:

Cor 2.15. Let $L = \mathfrak{sl}(2, F)$, and $V$ a finite dimensional $L$-module.

1. The eigenvalues of $h$ on $V$ are all integers, and each occur along its negative with equal number of times.

2. Suppose $V$ is decomposed into a direct sum of irreducible submodules: $V \simeq \sum_{m \in \mathbb{N}} a_m V(m)$.

Then the total number of irreducible summands is
\[
\sum_{m \in \mathbb{N}} a_m = \dim V_0 + \dim V_1;
\]
for $m \in \mathbb{N}$, the number of copies of $V(m)$ in $V$ is
\[ a_m = \dim V_m - \dim V_{m+2}. \]

Proof. (exercise) \hfill\(\square\)

In brief, given a finite dimension representation $\phi : L \rightarrow \mathfrak{gl}(V)$, the $h$-action on $V$ uniquely determines the weight spaces $V_\lambda$, their dimensions $\dim V_\lambda$, and the multiplicities $a_m$ of irreducible summands $V(m)$, in the representation $\phi$.

Ex. 1. In the natural representation $\phi_1 : L \rightarrow \mathfrak{gl}(F^2)$, the standard basis $B_1 := \{e_1, e_2\}$ of $F^2$ consists of eigenvectors of $\phi_1(h)$, and the matrix form $\phi_1(h) \approx \text{diag}(1, -1)$. Therefore, the $L$-module $F^2 \simeq V(1)$. 


2. In the adjoint representation \( \text{ad} : L \to gl(L) \), the basis \( B_2 := \{ x, h, y \} \) of \( L \) consists of eigenvectors of \( \text{ad} h \), and the matrix form \( \text{ad} h \approx \text{diag}(2, 0, -2) \). Therefore, the \( L \)-module \( L \simeq V(2) \).

3. Consider the tensor representation \( \phi_1 \otimes \text{ad} : L \to gl(F^2 \otimes L) \). Then \( F^2 \otimes L \) has a basis consisting of eigenvectors of \( (\phi_1 \otimes \text{ad})(h) \):

\[
B_1 \times B_2 = \{ e_1 \otimes x, e_1 \otimes h, e_1 \otimes y, e_2 \otimes x, e_2 \otimes h, e_2 \otimes y \},
\]

and the matrix form of the \( h \)-action w.r.t. this basis is

\[
(\phi_1 \otimes \text{ad})(h) \overset{B_1 \times B_2}{\approx} \text{diag}(3, 1, -1, 1, -1, -3).
\]

Then

\[
(\dim V_0, \dim V_1, \dim V_2, \dim V_3, \dim V_4, \dim V_5, \cdots) = (0, 2, 0, 1, 0, 0, \cdots).
\]

We get \( a_1 = 1 \), \( a_3 = 1 \), and the other \( a_k = 0 \). Hence the \( L \)-module \( F^2 \otimes L = V(1) \oplus V(3) \).