On Kostant's Partial Order on Hyperbolic Elements

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Let g be a complex or real $n \times n$ matrix. Define the following n-tuples (also viewed as diagonal matrices):

- $\vec{s} := (s_1, \cdots, s_n) \in (\mathbb{R}^+)^n$ singular values of g in descending order ($s_1 \geq s_2 \geq \cdots \geq s_n > 0$).

- $\vec{\lambda} := (\lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^\times)^n$ eigenvalues of g, with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| > 0$. Denote $|\vec{\lambda}| := (|\lambda_1|, \cdots, |\lambda_n|) \in (\mathbb{R}^+)^n$.

- $\vec{a} := (a_1, \cdots, a_n) = \mathrm{diag}(R) \in (\mathbb{R}^+)^n$ the diagonal part of the upper triangular matrix R obtained by the QR decomposition $g = QR$. Call $\vec{a}$ the a-component of g.

- $\vec{d} := \mathrm{diag}(g)$ the diagonal part of the matrix g.

Q: What are the relations of these scalars?

Answer 1: $(\vec{s}, |\vec{\lambda}|)$ and $(\vec{s}, \vec{a})$ obey multiplicative majorization relationships:

1. (H. Weyl [We], A. Horn [Ho], and C. Thompson [Th]) $\vec{s}$ and $\vec{\lambda}$ are the singular values and the eigenvalues of a matrix respectively if and only if $\vec{s}$ multiplicatively majorizes $|\vec{\lambda}|$:

$$\begin{aligned} s_1 s_2 \cdots s_k &\geq |\lambda_1 \lambda_2 \cdots \lambda_k|, \quad k = 1, \cdots, n-1, \\ s_1 s_2 \cdots s_n &= |\lambda_1 \lambda_2 \cdots \lambda_n|. \end{aligned}$$

2. (B. Kostant [Ko]) $\vec{s}$ and $\vec{a}$ are the singular values and the a-component of a matrix respectively iff $\vec{s}$ multiplicatively majorizes $\vec{a}$, that is, after rearranging the components of $\vec{a}$ in descending order: $a_1' \geq a_2' \geq \cdots \geq a_n' > 0$,

$$\begin{aligned} s_1 s_2 \cdots s_k &\geq a_1' a_2' \cdots a_k', \quad k = 1, \cdots, n-1, \\ s_1 s_2 \cdots s_n &= a_1' a_2' \cdots a_n'. \end{aligned}$$

3. Indeed, Kostant extended the multiplicative majorization relationships of the pairs $(\vec{s}, |\vec{\lambda}|)$ and $(\vec{s}, \vec{a})$ in terms of Kostant's partial order on connected real semisimple Lie groups.

   [Ko] Bertram Kostant, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. scient. Éc. Norm. Sup. (4e), **6** (1973) 413-455.

Remark. The other pairwise and triplewise relationships within the n-tuples $\vec{s}$, $\vec{\lambda}$, $\vec{a}$, and diag(U) (obtained from the Gelfand-Naimark decomposition $g = L\omega U$ of g), are also known (H. Huang, T. Y. Tam [HT]).

Answer 2: $(\vec{s}, \vec{d})$ obeys additive majorization relationship:

4. (I. Schur [Sch], A. Horn [Ho1]) $\vec{s} = \vec{\lambda}$ and $\vec{d}$ are the singular values and the diagonal of a Hermitian matrix iff $\vec{s}$ additively majorizes $\vec{d}$, that is, after rearranging the entries of $\vec{d}$ in descending order: $d_1 \geq d_2 \geq \cdots \geq d_n$,
$$\begin{aligned} s_1 + \cdots + s_k &\geq d_1 + \cdots + d_k, \quad k = 1, \cdots, n-1, \\ s_1 + \cdots + s_n &= d_1 + \cdots + d_n. \end{aligned}$$

5. (R.C.Thompson [Th1], F.Y.Sing [Si]) Let $\vec{s} \in (\mathbb{R}^+)^n$ and $\vec{d} \in \mathbb{C}^n$. Then there exists a complex $n \times n$ matrix with singular values $\vec{s}$ and diagonal entries $\vec{d}$ iff
$$\begin{aligned} \sum_{i=1}^{k} s_i &\geq \sum_{i=1}^{k} |d_i|, \qquad k = 1, \cdots, n \\ \sum_{i=1}^{n-1} s_i - s_n &\geq \sum_{i=1}^{n-1} |d_i| - |d_n|, \end{aligned}$$
after rearranging the entries of $\vec{s}$ and $\vec{d}$ in descending order w.r.t. modulus.

6. (Thompson) Let $\vec{s} \in (\mathbb{R}^+)^n$ and $\vec{d} \in \mathbb{R}^n$. Then there exists a real $n \times n$ matrix with positive determinant that has singular values $\vec{s}$ and diagonal $\vec{d}$ iff
$$\begin{aligned} \sum_{i=1}^{k} s_i &\geq \sum_{i=1}^{k} |d_i|, \quad k = 1, \cdots, n, \\ \sum_{i=1}^{n-1} s_i - s_n &\geq \sum_{i=1}^{n-1} |d_i| - d_n, \end{aligned}$$
and in addition, if the number of negative terms among $\vec{d}$ is odd,
$$\sum_{i=1}^{n-1} s_i - s_n \geq \sum_{i=1}^{n} |d_i|.$$

7. (T. Y. Tam, [T]) Thompson's results on the relations of $\vec{s}$ and $\vec{d}$ are special cases of [Ko, Thm 8.2] in terms of Kostants partial order on real semsimple Lie algebras. It is easy to see that Schur-Horn's result is also a special case of [Ko, Thm 8.2].

$\star$ The above majorization inequalities within several important n-tuples of a matrix are linked to Kostant's partial order on connected real semisimple Lie groups or Lie algebras.

A Lie group is a real or complex differential manifold with a group structure, where the multiplication and inverse operations are analytic.

The tangent space of a Lie group G at 1 constitutes a Lie algebra $\mathfrak{g}$ with a bilinear bracket operation $[,]$. $\mathfrak{g}$ is called the Lie algebra of G.

A connected real Lie group G is semisimple if its Lie algebra $\mathfrak{g}$ is semisimple, that is,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

where each $\mathfrak{g}_i$ is a simple ideals of $\mathfrak{g}$.

Many matrix groups are real semisimple Lie groups:

Ex. For $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, the groups

$$\mathsf{SL}_n(\mathbb{F}) = \{x \in \mathbb{F}^{n\times n} \mid \det x = 1\}$$

is connected real semisimple. Its Lie algebra is

$$\mathfrak{sl}_n(\mathbb{F}) = \{X \in \mathbb{F}^{n\times n} \mid \operatorname{tr} X = 0\}.$$

The bracket operation on the Lie algebra of a matrix group always has the form:

$$[X, Y] = XY - YX.$$

Ex. Some other examples of real connected semisimple Lie groups: (denote $J_{2n} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, $I_{p,q} := \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$.)

$$\begin{aligned}
\mathsf{SU}_n &= \{x \in \mathsf{SL}_n(\mathbb{C}) \mid x^* x = I_n\}, \\
\mathsf{SO}_n(\mathbb{C}) &= \{x \in \mathsf{SL}_n(\mathbb{C}) \mid x^T x = I_n\}, \\
\mathsf{Sp}_{2n}(\mathbb{C}) &= \{x \in \mathsf{SL}_{2n}(\mathbb{C}) \mid x^T J_{2n} x = J_{2n}\}, \\
\mathsf{SO}_n &= \{x \in \mathsf{SL}_n(\mathbb{R}) \mid x^T x = I_n\}, \\
\mathsf{Sp}_{2n} &= \{x \in \mathsf{SL}_{2n}(\mathbb{R}) \mid x^T J_{2n} x = J_{2n}\}, \\
\mathsf{SU}_{p,q} &= \{x \in \mathsf{SL}_{p+q}(\mathbb{C}) \mid x^* I_{p,q} x = I_{p,q}\}.
\end{aligned}$$

Let G be a connected real semisimple Lie group. Let $g \in G$. Similarly to matrix case, there are $\vec{s}$, $|\vec{\lambda}|$, and $\vec{a}$ for g in G obtained from three decompositions:

- $\vec{s}$ is obtained from the global Cartan decomposition of G:

  $$G = K \times P \quad \text{(a diffeomorphism onto).}$$

  . . . .

  When $G = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{R})$, the global Cartan decomposition of G is the usual matrix Cartan decomposition: Every determinant 1 matrix g can be written as $g = kp$, where k is unitary and $p = \exp H$ for a Hermitian matrix H.

- $\vec{a}$ is obtained from the Iwasawa decomposition of G:

  $$G = K \times A \times N \quad \text{(a diffeomorphism onto).}$$

  . . . .

  When $G = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{R})$, the Iwasawa decomposition of G is the matrix QR decomposition: Every determinant 1 matrix g can be written as $g = qr$, where q is unitary and r is upper triangular; write $a := \mathrm{diag}(r)$ and $n := a^{-1}r$ then $g = qan$.

- $|\vec{\lambda}|$ is obtained from the complete multiplicative Jordan decomposition (CMJD):

$$g = e(g)h(g)u(g).$$

. . . .

When $G = \mathrm{SL}_n(\mathbb{C})$, the CMJD of $g \in G$ is given as follow:

- If g is in a Jordan canonical form, then

$$\begin{aligned} e(g) &= \mathrm{diag}(g)\,|\mathrm{diag}(g)|^{-1}, \\ h(g) &= |\mathrm{diag}(g)|, \\ u(g) &= h(g)^{-1}e(g)^{-1}g. \end{aligned}$$

  So $e(g)$ is a diagonal matrix with modulus 1 entries, $h(g)$ is a diagonal matrix with positive real entries, $u(g)$ is a unit upper triangular matrix, and $e(g)$, $h(g)$ and $u(g)$ commute. Note that $h(g)$ gives the eigenvalue moduli $|\vec{\lambda}|$ of g.

- In general, $g = yJy^{-1}$ where $y \in G$ and J is a Jordan canonical form of g. Then

$$\begin{aligned} e(g) &= y\,e(J)\,y^{-1}, \\ h(g) &= y\,h(J)\,y^{-1}, \\ u(g) &= y\,u(J)\,y^{-1}. \end{aligned}$$

  Obviously, $e(g)$, $h(g)$ and $u(g)$ commute, and $h(g)$ gives the eigenvalue moduli $|\vec{\lambda}|$ of g.

To define Kostant's partial order on G, we need the following notations ([Kn]) and the definition of the CMJD ([Ko]).

$$\begin{aligned}
\mathfrak{g} &: \text{the Lie algebra of } G; \\
\theta &: \text{a Cartan involution of } \mathfrak{g},\ \theta^2 = 1 \in \mathrm{Aut}\,(\mathfrak{g}); \\
\mathfrak{k} &: +1 \text{ eigenspace of } \theta, \text{ a Lie subalgebra of } \mathfrak{g}; \\
\mathfrak{p} &: -1 \text{ eigenspace of } \theta, \text{ a subspace of } \mathfrak{g}; \\
\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} &: \text{the Cartan decomposition of } \mathfrak{g}; \\
\mathfrak{a}\ (\subseteq \mathfrak{p}) &: \text{a maximal abelian subspace of } \mathfrak{p}; \\
\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\gamma\in\Sigma} \mathfrak{g}_\gamma &: \text{restricted root space decomposition of } \mathfrak{g}; \\
\Sigma &= \Sigma^+ \cup (-\Sigma^+); \\
\Sigma^+ &: \text{the set of all positive restricted roots}; \\
\mathfrak{a}^+\ (\subseteq \mathfrak{a}) &: \text{the fundamental Weyl chamber in } \mathfrak{a}; \\
\mathfrak{n} = \bigoplus_{\gamma\in\Sigma^+} \mathfrak{g}_\gamma &: \text{a maximal nilpotent subalgebra}; \\
\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} &: \text{the Iwasawa decomposition of } \mathfrak{g}.
\end{aligned}$$

The $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{a}^+$, $\mathfrak{n}$ in Lie algebra $\mathfrak{g}$ can be lifted to their counterparts K, P, A, A^+, N in Lie group G by exponential map. So do the Cartan decomposition and the Iwasawa decomposition of $\mathfrak{g}$.

Define the Weyl group of G by:

$$W := N_K(\mathfrak{a})/Z_K(\mathfrak{a}),$$

where

$$\begin{aligned}
N_K(\mathfrak{a}) &:= \{k \in K \mid \mathrm{Ad}k(\mathfrak{a}) \subseteq \mathfrak{a}\}, \\
Z_K(\mathfrak{a}) &:= \{k \in K \mid \mathrm{Ad}k(X) = X \text{ for all } X \in \mathfrak{a}\}.
\end{aligned}$$

The CMJD: (Kostant [Ko]) Let G be a connected real semisimple Lie group. Every $g \in G$ has the unique complete multiplicative Jordan decomposition (CMJD) $g = ehu$, where $e, h, u \in G$ satisfy that:

- ⋆ e is elliptic: Ad$(e) \in$ Aut $(\mathfrak{g})$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1;

- ⋆ h is hyperbolic: $h = \exp X$ where $X \in \mathfrak{g}$ is real semisimple, i.e. ad$(X) \in$ End $(\mathfrak{g})$ is diagonalizable over $\mathbb{R}$ with real eigenvalues;

- ⋆ u is unipotent: $u = \exp X$ where $X \in \mathfrak{g}$ is nilpotent, i.e. ad$(X) \in$ End $(\mathfrak{g})$ is nilpotent.

- ⋆ e, h, and u mutually commute (thus they also commute with g).

We write $g = e(g)h(g)u(g)$.

Basic properties of CMJD:

1. The CMJDs are preserved by conjugations & direct sums.

2. The CMJDs are preserved by group homomorphisms. Let $\phi : G \to H$ be a homomorphism of connected real semisimple Lie groups G and H. If $g = ehu$ is the CMJD of g in G, then $\phi(g) = \phi(e)\phi(h)\phi(u)$ is the CMJD of $\phi(g)$ in H.

   In particular, the CMJD of g in a matrix group $G \subseteq \mathrm{SL}_n(\mathbb{C})$ is identical with the CMJD of g in $\mathrm{SL}_n(\mathbb{C})$.

3. In the CMJD $g = ehu$, the hyperbolic component h is G-conjugate to a unique element

$$|\vec{\lambda}|(g) \in A_+,$$

   which is the counterpart of the eigenvalue moduli $|\vec{\lambda}|$ of a matrix.

4. If $g = kp$ ($k \in K$, $p \in P$) is the global Cartan decomposition of $g \in G$, then

$$\vec{s}(g) := |\vec{\lambda}|(p) \in A_+$$

   is the counterpart of the singular values $\vec{s}$ of a matrix.

Kostant's partial order ([Ko]).

Let g be any element of connected real semisimple Lie group G.

$$\begin{aligned} g \in G \quad &\Rightarrow \quad CMJD: \ g = ehu \\ &\Rightarrow \quad h \text{ is } G\text{-conjugate to } |\vec{\lambda}|(g) \in A_+ \\ &\Rightarrow \quad \text{denote } \mathcal{A}(g) := \exp\left\{\text{conv}\left[W \cdot \log|\vec{\lambda}|(g)\right]\right\} \end{aligned}$$

Roughly speaking, $\mathcal{A}(g)$ is the multiplicative convex hull of the Weyl group orbit of $|\vec{\lambda}|(g) \in A_+$.

The Kostant's partial order in G is defined as:

$$g_1 \succeq g_2 \iff \mathcal{A}(g_1) \supseteq \mathcal{A}(g_2).$$

As mentioned early, Kostant's partial order determines the relations of $\vec{s}$ and $|\vec{\lambda}|$ and $\vec{s}$ and $\vec{a}$, etc.

Theorem 1 (Kostant [Ko]). *Given $\vec{s}, |\vec{\lambda}| \in A_+$, there exists $g \in G$ satisfying $\vec{s}(g) = \vec{s}$ and $|\vec{\lambda}|(g) = |\vec{\lambda}|$ iff $\vec{s} \succeq |\vec{\lambda}|$.*

Similarly, given $\vec{s}, \vec{a} \in A_+$, there exists $g \in G$ satisfying $\vec{s}(g) = \vec{s}$ and $\vec{a}(g) = \vec{a}$ iff $\vec{s} \succeq \vec{a}$.

Theorem 2 (Kostant [Ko]). *Let $g_1, g_2 \in G$. Then $g_1 \succeq g_2$ iff the spectral radii*

$$|\pi(g_1)| \geq |\pi(g_2)|$$

for all finite dimensional representation π of G. ($|\pi(g)|$ is the maximal eigenvalue modulus of the matrix $\pi(g)$).

Together with the k-compound representations of special linear groups, Theorems 1 and 2 imply the multiplicative majorization relationships of the pairs $(\vec{s}, |\vec{\lambda}|)$ and $(\vec{s}, \vec{a})$ for a matrix $g \in \mathrm{SL}_n(\mathbb{C})$.

Now consider the set of hyperbolic elements:

$\{h \in G \mid h = \exp X,\ \text{ad}X \text{ is real diagonalizable}\}$

The hyperbolic elements in $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{R})$ are those matrices diagonalizable with positive real eigenvalues.

Theorem 3 (Kostant [Ko]). *Let $h_1, h_2 \in G$ be hyperbolic. If $h_1 \succeq h_2$, then $\chi_\pi(h_1) \geq \chi_\pi(h_2)$ for all finite dimensional representations π of G, where $\chi_\pi(h) = \mathrm{tr}\,(\pi(h))$ denotes the character of π.*

Kostant asked if the converse of Theorem 3 holds or not. It was answered affirmatively recently (H. Huang and S. Kim [HK]). So Theorem 3 can be improved as follow:

Theorem 4 (Huang & Kim [HK]). *Let h_1, h_2 be hyperbolic elements in G. Then $h_1 \succeq h_2$ iff the character values $\chi_\pi(h_1) \geq \chi_\pi(h_2)$ for all finite dimensional representations π of G.*

If $g = ehu$ is the CMJD of any $g \in G$, then $\pi(g) = \pi(e)\pi(h)\pi(u)$ is the CMJD of $\pi(g)$ in the matrix group $\pi(G)$. Conjugating $\pi(g)$ to its Jordan canonical form, we see that the eigenvalue moduli of $\pi(g)$, denoted by $\lambda_\pi^1(g) \geq \cdots \geq \lambda_\pi^n(g) > 0$, are precisely the eigenvalues of $\pi(h)$. Denote

$$|\chi_\pi|(g) := \lambda_\pi^1(g) + \cdots + \lambda_\pi^n(g) = \chi_\pi(h).$$

Theorem 4 is equivalent to that:

Corollary 5 (Huang & Kim [HK]). *Suppose $g_1, g_2 \in G$. Then $g_1 \succeq g_2$ iff $|\chi_\pi|(g_1) \geq |\chi_\pi|(g_2)$ for all finite dimensional representations π of G.*

More nice relations between Kostant's partial order "$\succeq$" on G and the multiplicative and additive majorizations of the eigenvalue moduli in finite dimensional representations of G are given below:

Theorem 6 (Huang & Kim [HK]). *Let G be a connected real semisimple Lie group and $g_1, g_2 \in G$.*

1. *[Ko] If $g_1 \succeq g_2$ in G, then for every finite dimensional representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$, the following inequalities hold for $k = 1, \cdots, n$:*

$$\prod_{i=1}^{k} \lambda_\pi^i(g_1) \ge \prod_{i=1}^{k} \lambda_\pi^i(g_2), \quad \sum_{i=1}^{k} \lambda_\pi^i(g_1) \ge \sum_{i=1}^{k} \lambda_\pi^i(g_2).$$

2. *Fix $k \in \mathbb{Z}^+$. If for every finite dimensional representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$ with $n \ge k$ we have*

$$\sum_{i=1}^{k} \lambda_\pi^i(g_1) \ge \sum_{i=1}^{k} \lambda_\pi^i(g_2),$$

   then $g_1 \succeq g_2$ in G.

3. *Fix $k \in \mathbb{Z}^+$. If for every finite dimensional representation $\pi : G \to \mathrm{GL}_n(\mathbb{C})$ with $n \ge k$ we have*

$$\prod_{i=1}^{k} \lambda_\pi^i(g_1) \ge \prod_{i=1}^{k} \lambda_\pi^i(g_2),$$

   then $g_1 \succeq g_2$ in G.

Ex. Let $G := \mathrm{SL}_n(\mathbb{C})$ and $g \in G$. We know that $\vec{s}$ multiplicatively majorizes $|\vec{\lambda}|$ and $\vec{a}$. Indeed, viewed as diagonal matrices, $\vec{s} \succeq |\vec{\lambda}|$ and $\vec{s} \succeq \vec{a}$ in G. So for any $k \in \mathbb{Z}^+$ and any finite dimensional representation π of G, the sum or product of the largest k eigenvalues of $\pi(\vec{s})$ is greater than or equal to its counterparts of $\pi(|\vec{\lambda}|)$ and $\pi(\vec{a})$.

References:

[HK] Huajun Huang and Sangjib Kim, *On Kostant's partial order on hyperbolic elements*, Linear and Multilinear Algebra, to appear.

[HT] Huajun Huang and Tin-Yau Tam, *On the Gelfand-Naimark decomposition of a nonsingular matrix*, Linear and Multilinear Algebra, **58** (2010) 27-43.

[Ho] Alfred Horn, *On the eigenvalues of a matrix with prescribed singular values*, Proceedings of the Amer. Math. Soc., **5** (1954) 4-7.

[Ho1] A. Horn, *Doubly stochastic matrices and the diagonal of a rotation matrix*, Amer. J. Math., **76** (1954) 620-630.

[Kn] Anthony W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996.

[Ko] Bertram Kostant, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. scient. Éc. Norm. Sup. (4e), **6** (1973) 413-455.

[Sch] I. Schur, *Über eine Klasse von Mittelbildungen mit Anwendungen auf der Determinantentheorie*, Sitzungsberichte der Berlinear Mathematischen Gesellschaft, **22** (1923) 9-20.

[Si] F. Y. Sing, *Some results on matrices with prescribed digonal elements and singular values*, Canad. Math. Bulletin, **19** (1976) 89-92.

[T] T. Y. Tam, *a Lie theoretical approach of Thompson's theorems of singular values-diagonal elements and some related results*, J. of London Math. Soc. (2), **60** (1999) 431-448.

[Th] Colin Thompson, *Inequalities and partial orders on matrix spaces*, Indiana Univ. M. th. J., **27** (1971) 469-480.

[Th1] R. C. Thompson, *Singular values, diagonal elements and convexity*, SIAM J. Appl. Math. **32** (1977) 39-63.

[We] Hermann Weyl, *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad. Sci. U. S. A. **35** (1949) 408–411.

Thank you!!!